1968-06

Methods of Analysis For Linear Systems with Time Varying Parameters.

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METHODS OF ANALYSIS FOR LINEAR SYSTEMS WITH TIME VARYING PARAMETERS

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by

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Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
June 1968

ABSTRACT

This thesis presents a comparative study of frequency domain and state variable (time domain) methods for the solution of linear time varying systems. Stability criteria for linear feedback systems, where the time variation is limited to the feedback loop, are considered. In comparing state variable and frequency domain solutions, the conclusion is reached that state variable methods are more useful. Experimental computer results are presented which indicate that, for the feedback system considered, sufficient conditions for stability can be refined by considering the system to be dual mode (a stable mode and an unstable mode), and by considering which mode predominates.

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LIST OF SYMBOLS AND ABBREVIATIONS

$\underline{A}$	Matrix
$\underline{A}(t)$	Time varying matrix
$\underline{\phi}(t,t_o)$	State transition matrix
$\underline{\phi}(\underline{A}(KT),T)$	Discrete state transition matrix for fixed matrix $\underline{A}(KT)$ and period T.
$x(t)$	Time varying quantity (scalar or vector)
$x^T(t)$	Transpose of vector $x(t)$
$X(\cdot)$	Frequency transform of $x(t)$
λ	Generalized frequency transform variable
s	Laplace frequency transform variable
ω	Fourier frequency transform variable
C_λ	Integration contour in λ-plane
$K(\lambda,t)$	Direct kernal for generalized transform
$K(\lambda,t)$	Inverse kernal for generalized transform
$L(p,t)$	Linear differential operator
p	d/dt (when used with linear differential operator)
$\triangleq$	Denotes
$\equiv$	Is defined as
$\lvert\cdot\rvert$	Magnitude of
$\lVert\cdot\rVert$	Norm of
det	Determinate of
tr	Trace of (sum of diagonal terms of a matrix)
ASIL	Asymptotically stable in the large
SIL	Stable in the large
S_u	Mode stability ratio

Re	Real part of
Im	Imaginary part of

ACKNOWLEDGEMENT

The author wishes to express his appreciation to Professor Sydney R. Parker for guidance and suggestions made in the preparation of this thesis.

1. INTRODUCTION

A. Background. Almost all physical systems are time varying. In many cases, the time variation is sufficiently slow so that the simplification of assuming time invariance is valid. Prior to 1950, control engineering theory was primarily limited to the analysis and synthesis of systems with fixed parameters. Since 1950, a growing number of papers [12,20] have appeared which discuss systems with varying parameters. The principal approaches to time varying system study generally fall into one of the following categories:

a. Generalization of frequency domain methods to time varying systems,

b. Time domain analysis of time varying systems,

c. Time varying systems as solutions to optiminal control problems.

d. Extensions of stability criteria for fixed systems to time varying systems; or the establishment of simple stability criteria for time varying systems.

In this thesis, frequency and time domain approaches to the solution of time varying systems are reviewed, and stability criteria for a specific class of time varying feedback systems considered. The discussion is limited to linear systems.

B. Linear Systems. The basic requirements for a linear system [7] are superposition (additivity) and homogeneity. Homogeneity implies that the zero-state response of a system to an input of magnitude K is just K times the zero-state response to the same input of unit magnitude [7]. Superposition implies that the zero-state response to the sum of several inputs is equal to the sum of the zero-state responses of the system to the same inputs acting independently. If a system is both homogeneous and additive, then the zero-state response of the system to a given input can be determined by decomposing the input into a set of basis functions, determining the response to each function, and summing the responses. This process can be generalized (for example; when the input basis is a set of impulse function) by the superposition integral.

A basic property of all linear systems is that the response of the system [7] can be separated into terms related to the initial (zero) state values and into terms related to the system input.

2. TRANSFORM METHODS

A. Introduction. The descriptive model which views a transform as the projection of a signal vector onto an m-dimension orthogonal coordinate system has been used more frequently in the study of communication systems than in the study of control systems. From this point of view, amplification is a linear stretching of the spectrum space and modulation is a warping of spectrum space. The latter concept is of use in the discussion of time varying systems. Modulation does not occur in linear time invariant systems. However, even in simple time varying feedback systems, such as figure 2-1, the effect of modulation is the key to the analysis of the system. The modulation effects of the feedback, resulting from the product term $a(t)y(t)$, is discussed in Chapter 5. In this chapter the following types of spectra are considered:

a. The sinusoidal steady state frequency spectrum ($j\omega$),

b. The complex frequency spectrum (s),

c. The generalized frequency spectrum (λ).

The sinusoidal frequency and complex frequency spectra are obtained using the Fourier and Laplace transforms respectively. The generalized frequency spectrum is obtained using the following general transform pair [1]:

$$f(t) = \frac{1}{2\pi j} \int_{C_\lambda} F(\lambda)\, K^{-1}(\lambda, t)\, d\lambda \qquad (1)$$

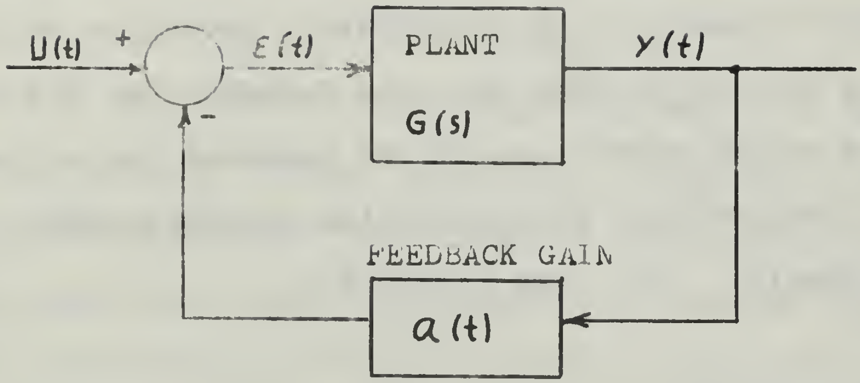
and:

$$F(\lambda) = \int_a^b f(t)\ K(\lambda,t)dt \qquad (2)$$

where:
a. $f(t)$ denotes a time function,
b. $F(\lambda)$ denotes the general frequency domain function,
c. $K(\lambda,t)$ denotes the direct transform kernal,
d. $K^{-1}(\lambda,t)$ denotes the inverse transform kernal,
e. C_λ is an appropriate contour in the λ-plane which encloses all of the poles of the integrand.

The kernal is the expansion weighting function. In the Laplace transform, the kernal is exp(-st) where the variable, s, corresponds to the variable λ in the generalized transform. The generalized transform becomes necessary in the analysis of time varying control systems because the outputs of these systems cannot usually be represented as linear combinations of simple time constants (exponentials). In the discussion which follows, the response of a simple time varying feedback system is considered first. Then the generalized transform, the compatible (time invariant) system function, the incompatible (time varying) system function, and transfer functions based on the incompatible system function are discussed in order.

B. Simple Time Constant Response. In the system of figure 2-1, the output of a time invariant system is modulated and used as a negative feedback signal. This section



$$G(s) = \frac{Q(s)}{P(s)}$$

where: u(t) is the system input

ε(t) is the error function

G(s) is a fixed parameter plant

y(t) is the system output

a(t) is the linear time varying gain

Q(s) is the numerator polynominal of G(s)

P(s) is the denominator polynominal of G(s)

Figure 2-1: <u>Example of a Simple Time Varying Feedback System</u>

is concerned with the form that the output, $y(t)$, assumes. In the first example it is assumed that the feedback gain is a constant, A_o, and the input is an impulse function. Now, if $G(s)$ is assumed to have a first order pole and no zeros, it is known that the output, $y(t)$, will be the simple time constant decay: $K\exp(-\lambda t)$. The purpose of the first example is to illustrate a procedure which will be used shortly to show that the response for the time varying system cannot usually be expected to involve the linear combinations of simple time constant terms.

Example: 2-1. See figure 2-1

Given:

$$G(s) = \frac{1}{s+p}$$

$$U(t) = \delta(t)$$

$$a(t) = A_o$$

Assume:

$$y(t) = Ke^{-\lambda t} \quad \text{or}$$

$$y(s) = \frac{K}{s+\lambda}$$

where K and λ are to be determined.

From figure 2-1:

$$Y(s) + A_o\ G(s)y(s) = G(s) \tag{1}$$

Substituting the assumed values of $G(s)$ and $Y(s)$ into equation (1) yields:

$$\frac{K}{s+\lambda} + \frac{A_oK}{s+\lambda} \cdot \frac{1}{s+p} = \frac{1}{s+p} \tag{2}$$

Equation (2) may be simplified to:

$$Ks + Kp + A_oK = s+\lambda \tag{3}$$

Equation (3) reduces to:

$$Ks + K(p + A_o) = s + \lambda \tag{4}$$

By equating the coefficients of like terms:

$$K = 1$$

and: (5)

$$A_o + p = \lambda$$

Hence, the output, y(t), is exponential.

Now, applying the procedure of example 2-1 to a time varying feedback system, where $a(t) = \exp(-\alpha t)$, it is demonstrated that the output cannot be a simple time constant combination when the input is an impulse function. While the impulse function is used as the input in this example, the procedure may be extended to other inputs.

Example 2-1: See figure 2-1.

Given:

$$G(s) = \frac{1}{s+p}$$

$$U(t) = \delta(t)$$

$$a(t) = e^{-\alpha t}$$

Assume:

$$y(t) = K\,e^{-\lambda t} \quad \text{or}$$

$$Y(s) = \frac{K}{s+\lambda}$$

where K and λ are to be determined.

From figure 2-1, where "$*$" denotes complex convolution*

$$Y(s) + G(s)\left[Y(s) \quad \frac{1}{s+\alpha}\right] = G(s) \tag{6}$$

Because of the convolution properties* of exponentials, equation (6) reduces to

$$Y(s) + G(s)\,Y(s+\alpha) \quad = \quad G(s) \tag{7}$$

Substituting the assumed values of G(s) and Y(s) into equation (7) yields:

$$\frac{K}{s+\lambda} + \frac{1}{s+p} \cdot \frac{K}{s+\alpha+\lambda} = \frac{1}{s+p} \tag{8}$$

Equation (8) may be simplified to

$$K(s+\alpha+\lambda)(s+p) + K(s+\lambda) = (s+\lambda)(s+\alpha+\lambda) \tag{9}$$

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* In general [7] $\mathcal{L}[f_1(t)\ f_2(t)] = F_1(s)*F_2(s)$

$$= \frac{1}{2\pi j}\int_{\sigma_1-j\omega}^{\sigma_1+j\omega} F_1(\lambda)\ F_2(s-\lambda)\,d\lambda$$

where the integration is carried out along the vertical line $\text{Re}(s) = \sigma_1$ with σ_1 and s constrained such that:

I. $\sigma_1 > \alpha_1$ (abscissa of convergence of $f_1(t)$)

II. $\text{Re}\ s-\sigma_1 > \alpha_2$ (abscissa of convergence of $f_2(t)$)

If $f_2(t) = \exp(-\alpha t)$, the convolution [8] is given by:

$F_1(s) \quad \frac{1}{s+\alpha} = F_1(s+\alpha)$

Carrying out the multiplications of equation (9) and collecting coefficients of like powers of s on both sides of the equation yields:

$$Ks^2 + K(1+\alpha+\lambda+p)s + K(\alpha p+\lambda p+\lambda) = s^2 +(\alpha+2\lambda)s + \lambda(\alpha+\lambda) \quad (10)$$

Equating the coefficients of like powers of s yields:

$$\lambda = 1 + p \quad (11)$$

$$K = 1 \quad (11a)$$

$$\lambda^2 + \lambda\ (\alpha-p-1) + \alpha p = 0 \quad (12)$$

In general, solution of equations (11) and (12) are incompatible. Hence, a simple time constant solution is not possible. This result is consistent with the view that the time varying feedback is a modulation component. Hence, modulation must appear in the output unless the fixed plant, G(s), filters the modulation.

It is possible to construct a time varying system with a simple time constant output by means of a suitable choice of the fixed plant, G(s), so that the modulation terms are filtered out of the output.

Example 2-3: See figure 2-1.

Given:

$$y(s) = \frac{K}{s+\lambda}$$

$$u(t) = \delta(t)$$

$$a(t) = e^{-\alpha t}$$

Determine: G(s) such that the output is $\exp(-\lambda t)$.

From figure 2-1:

$$Y(s) + G(s)\left[Y(s) * \frac{1}{s+\alpha}\right] = G(s) \quad (13)$$

Because of the properties of the exponential, equation (13) reduces to:

$$Y(s) + G(s)\ Y(s+\alpha) = G(s) \tag{14}$$

Substituting the assumed values of Y(s) into equation (14) yields:

$$G(s) = \frac{\frac{K}{s+\lambda}}{1-\frac{K}{s+\lambda+\alpha}} = \frac{K(s+\alpha+\lambda)}{(s+\lambda)(s+\lambda+\alpha-K)} \tag{15}$$

If G(s) is to be asymptotically stable in the large, then it is required that:

$$\lambda > 0 \tag{16}$$

$$\lambda + \alpha - K > 0 \tag{17}$$

For example let:

$$\lambda = 4 \tag{18}$$

$$\alpha = 2 \tag{19}$$

$$K = 1 \tag{20}$$

Substituting values (18), (19), and (20) into constraint (17) yields:

$$\lambda + \alpha - K = 5 > 0 \tag{21}$$

and G(s) becomes:

$$G(s) = \frac{s+6}{(s+4)(s+5)} \tag{22}$$

This particular G(s) has the necessary cancellation properties for the given feedback modulation and input. If either is changed, the output will no longer be a simple time constant. To illustrate this dependence, let the feedback gain be changed to exp(-3t), then, with K=1, equation (14) becomes:

$$Y(s) + G(s)\ Y(s) * \frac{1}{s+3} = G(s) \tag{23}$$

Performing the indicated multiplications and clearing fractions yields:

$$\frac{s^3 + (13+\lambda)s^2 + (53+10\lambda)s + 60 + 26\lambda}{s^3 + (2\lambda+9)s^2 + (\lambda^2+15\lambda+18) + \lambda^2 + 18\lambda} \tag{24}$$

Equating like powers of s on both sides of equation (24) yields:

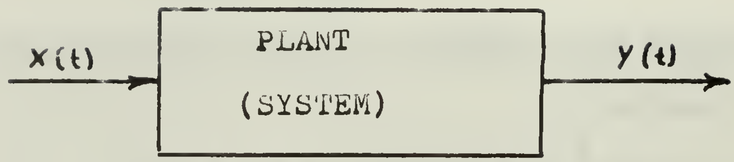
$$\begin{aligned} 53 + 10\lambda &= \lambda^2 + 15\lambda + 18 \\ 60 + 26\lambda &= \lambda^2 + 18\lambda \\ 13 + \lambda &= 2\lambda + 9 \end{aligned} \tag{25}$$

Since there is no λ^2 term on the left hand side of equation set (25), this set of equations is incompatible and the system output is no longer a simple time constant.

C. Generalized Transforms. For the plant of figure 2-2, it is assumed that the input/output relation is known and given by:

$$L_1(p,t)\ y(t) = L_2(p,t)\ x(t) \tag{1}$$

The explanation of the symbols in equation (1) is shown on figure 2-2. The input/output relation is assumed to represent a physical plant and is causal(non-anticipatory). For convenience, it is also assumed that the initial time is zero. If the initial time is not zero, the output is not simply a function of the delay time. If the left hand side of equation (1) is a first order differential equation, an integrating factor solution is possible [3,9] as follows:



$$L_1(p,t)\,Y(t) = L_2(p,t)\,X(t)$$

$$X(t) \triangleq \text{INPUT}$$

$$Y(t) \triangleq \text{OUTPUT}$$

$$L_1(p,t) \triangleq a_N(t)p^N + \cdots a_1(t)p + a_0(t)$$

$$L_2(p,t) \triangleq b_m(t)p^m + \cdots b_1(t)p + b_0(t)$$

$$p \triangleq \frac{d}{dt}$$

Figure 2-2: Basic Plant (System)

Example 2-4:

Given: $L_1(p,t) = a_1(t)p + a_o(t)$

Determine: Output $y(t)$

Substituting $L_1(p,t)$ into equation (1) yields:

$$a_1(t)p\ y(t) + a_o(t)\ y(t) = L_2(p,t)\ x(t) \tag{2}$$

Assuming that $a_1(t)$ is not equal to zero at any time greater than zero, it is possible to divide by $a_1(t)$ giving:

$$p\ y(t) + a(t)\ y(t) = \frac{1}{a_1(t)}\ L_2(p,t)\ x(t) \tag{3}$$

where:

$$a(t) = \frac{a_o(t)}{a_1(t)}$$

Equation (3) is of the general form:

$$\frac{dy(t)}{dt} + P(t)\ y(t) = Q(t) \tag{4}$$

For equation (4), the partial differential equation for the integrating factor [9], u, is:

$$[P(t)\ y(t) - Q(t)]\ \frac{\delta u}{\delta y} - \frac{\delta u}{\delta t} + u\ P(t) = 0 \tag{5}$$

If the integrating factor, u, is assumed to be a function of time only, then equation (5) reduces to:

$$\frac{\delta u}{\delta t} = u\ P(t) \tag{6}$$

The particular integral solution to equation (6) is:

$$u(t) = K\exp\left[\int^{t} P(\tau)\,d\tau\right] \tag{7}$$

Applying equation (7) to equation (3) to obtain the integrating factor yields:

$$u(t) = K\exp\left[\int^{t} a(\tau)\,d\tau\right] \tag{8}$$

If both sides of equation (2) are multiplied by the integrating factor, then

$$K\exp\left[\int^t a(\tau)\,d\tau\right]\left\{py(t)+a(t)y(t)\right\}=K\exp\left[\int^t a(\tau)\,d\tau\right]\frac{L_2(p,t)x(t)}{a_1(t)} \tag{9}$$

Equation (9) may be re-written as an exact differential equation on the left hand side giving:

$$\frac{d}{dt}\left\{y(t)\ \exp\left[\int^t a(\tau)\,d\tau\right]\right\} = \exp\left[\int^t a(\tau)\,d\tau\right]\frac{L_2(p,t)x(t)}{a_1(t)} \tag{10}$$

Direct integration yields:

$$y(t)\ \exp\left[\int^t a(\tau)\,d\tau\right] = \int^t \exp\left[\int^\tau a(\tau)\,d\tau\right]\left\{\frac{L_2(p,\tau)x(\tau)}{a_1(\tau)}\right\}d\tau + \text{constant} \tag{11}$$

If $\exp\left[\int^t a(\tau)\,d\tau\right] \neq 0$ for $t \geq 0$.

$$y(t) \geq \exp\left[-\int^t a(\tau)\,d\tau\right]\left\{\int^t \exp\left[\int^\tau a(\lambda)\,d\tau\right]\left[\frac{L_2(p,\tau)x(\tau)}{a_1(\tau)}\right]d\tau + \text{constant}\right\} \tag{12}$$

It is clear form equation (12) that if a(t) is a constant, then the integrating factor is always a simple exponential, exp(at). If a(t) is not a constant, then the form of the integrating factor is not generally a simple exponential, but a function of a(t)*. It is of interest to look at two

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* If

$$a(t) = \frac{2}{t}\ ,\ \exp\left[\int^t \frac{2d\tau}{\tau}\right] = \exp\left[\ln t^2\right] = t^2$$

The integrating factor is an indefinite integral [3,9].

examples where the input is assumed to be $\exp(-\lambda t)$, where λ is a complex constant. In the first example, the feedback gain will be assumed constant. In the second example, the feedback gain will be assumed to be a function of time.

Example 2-5:

Given: $a(t) = A_o$ $\qquad L_2(p,t) = L_2(p)$

$x(t) = \exp(-\lambda t)$

Determine: the output, $y(t)$

Substituting the assumed values of feedback gain and input into equation (3) yields:

$$Py(t) + A_o y(t) = L_2(p,t)\; x(t) = L_2(p)\; e^{-\lambda t} = e^{-\lambda t} L_2(-\lambda) \tag{13}$$

The solution of equation (13) from equation (12) is:

$$y(t) = \exp\left[A_o t\right]\left\{\int_o^t \exp[A_o\tau]\; e^{-\lambda\tau}\; L_2(-\lambda)\,d\tau\right\} \tag{14}$$

Combining exponential terms within the braces gives:

$$y(t) = \exp\left[-A_o t\right]\left\{\int_o^t \exp[(A_o-\lambda)\tau]\; L_2(-\lambda)\,d\tau\right\} \tag{15}$$

Performing the indicated integration yields:

$$y(t) = \exp\left[-A_o t\right]\left\{\frac{L_2(-\lambda)\left[\exp[(A_o-\lambda)t - 1\right]}{A_o-\lambda}\right\} \tag{16}$$

Combining the exponential terms in equation (16) gives:

$$y(t) = \frac{L_2(-\lambda)\;\exp\;[-\lambda t]}{A_o-\lambda} - \frac{L_2(-\lambda)\;\exp\;[-A_o t]}{A_o-\lambda} \tag{17}$$

In example 2-5, the steady state output, $y(t)$, is of the same form as the input but of different magnitude. A function whose shape is not changed between the input and output of the system is called a <u>characteristic function</u> of the system [2]. The fact that the simple exponential is a characteristic function for all time invariant systems is the basis of Fourier and Laplace analysis. In the previous section it was shown that the output of a time varying feedback system to an impulse input will generally not be in the form of simple time constants. In the next example, it will be demonstrated that the exponential function does not preserve its form between the input and output of a time varying system.

Example 2-6:

Let: $py(t) + e^{\alpha t}\, y(t) = L_2(p)\, e^{\lambda t} = e^{\lambda t}\, L(\lambda)$

where: $x(t) = e^{\lambda t}$

$a(t) = e^{\alpha t}$ (in equation 3)

Then, from equation (12):

$$y(t) = \exp\left[-\int^{t} e^{\alpha\tau} d\tau\right]\left\{\int_{0}^{t} \exp\left[\int^{\tau} e^{\alpha\rho} d\rho\right] e^{\lambda\tau}\, L_2(\tau)\, d\tau\right\} \tag{18}$$

After integrating the function in the exponential, equation (18) yields:

$$y(t) = \exp\left[-\frac{e^{\alpha t}}{\alpha}\right]\left\{\int_{0}^{t} \exp\left[\frac{e^{\alpha\tau}}{\alpha} + \lambda\tau\right] L_2(\lambda)\, d\tau\right\} \tag{19}$$

Hence, $y(t)$ does not have the same steady state form as $x(t)$.

Using the integrating factor approach, the output, y(t), of first order systems can usually be obtained. For some second order systems explicit solutions exist [9]. However, for most second and higher order systems, an explicit solution of the differential equation is not always possible. For time invariant systems a similar problem is simplified by transforming the differential equation into an algebric expression with either the Laplace or Fourier transform. If the Laplace transform:

$$F(s) = \int_0^\infty f(t)\ e^{-st}\ dt \tag{19}$$

is applied to the differential equation (1), the result is a convolution integral equation. Since:

$$\mathcal{L}\left[L_1(p,t)\ y(t)\right] = \mathcal{L}\left[L_2(p,t)\ x(t)\right] \tag{20}$$

and since*

$$\mathcal{L}\left[f_1(t)\ f_2(t)\right] = F_1(s) * F_2(s) \tag{21}$$

equation (20) becomes:

$$\mathcal{L}\left[L_1(p,t)\right] * y(s) = \mathcal{L}\left[L_2(p,t)\right] * x(s) \tag{22}$$

Since equation (22) is generally not algebric, the Laplace transform does not simplify the problem.

In equation (19), the term exp(-st) is called the kernal. For time invariant systems, exp(-st) is a <u>compatible kernal</u> - compatible in the sense that exp(-st) is a characteristic function for a time invariant system.

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* See section 2-B

Hence, if the input signal is decomposed into a spectrum in terms of the compatible kernal*, the spectrum of the output will also be in terms of the compatible kernal and can be obtained by algebric operations on each of the components of the input spectrum. Hopefully, in generalizing the transform procedure to time varying systems, it will be possible to determine a compatible kernal such that the generalized transform is:

$$F(\lambda) = \int_a^b f(t)\, K_c(\lambda,t)\, dt \tag{23}$$

where: $K_c(\lambda,t)$ is the compatible kernal

λ is the generalized frequency term, i.e. the parameter of the spectrum expansion.

Additionally, if equation (23) exists then it may be possible to write the generalized inverse transform [1] as:

$$f(t) = \frac{1}{2\pi j} \int_{C_\lambda} F(\lambda)\, K_c^{-1}(\lambda,t)\, d\tau \tag{24}$$

where: $K_c^{-1}(\lambda,t)$ is the inverse compatible kernal

C_λ is an integration contour in the λ-plane

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* The compatible kernal need not be unique. For example, either the Laplace or the Fourier kernals are compatible for time invariant systems.

Usually the inverse compatible kernal is the reciprocal of the compatible kernal, but this need not always be true [10].

Assuming that the compatible kernal has been determined for equation (23), then the transform of the derivative

$$\text{Transform}\left[\frac{df(t)}{dt}\right] = \int_a^b f'(t)\, K_c(\lambda,t)\, dt \tag{25}$$

is evaluated by integration by parts [8] to obtain:

$$\int_a^b f'(t)\, K_c(\lambda,t)\, dt = - \int_a^b f(t)\, K_c'(\lambda,t)\, dt + f(b)K_c(\lambda,b) - f(a)K_c(\lambda,a) \tag{26}$$

Since equation (19) and equation (23) represent only a means of obtaining a spectral expansion in terms of the selected kernal, it is necessary to return to the system input/output relation to determine which function(s) to use as compatible kernals. In order to simplify the determination of the compatible kernal, it is desirable to define a compatible system function. This definition is considered in the next section.

D. Compatible System Function. In this section the steady state value of the compatible system function due to L. A. Zadeh is considered. The approach used is due to Johnson and Kilmer [1]. In the following, the terms system function or compatible system function will mean the steady state value of the Zadeh system function. The compatible system function is defined by:

$$H(\lambda) \equiv \frac{y(t)}{x(t)} \quad x(t) = K_c^{-1}(\lambda, t) \text{ or } K_c(\lambda, t) \tag{1}$$

where: $H(\lambda)$ is the steady state response for the selected characteristic function (compatible kernal)

$K_c^{-1}(\lambda, t)$ is the inverse compatible kernal

$K_c\ (\lambda, t)$ is the compatible kernal.

The kernal input is a function of both the generalized frequency, λ, and time*. Since both the compatible kernal and the inverse compatible kernal** are characteristic functions, either may be used as the generating input for $H(\lambda)$. However, the resulting $H(\lambda)$ will be different for each case. In the following the inverse compatible kernal will be used as the input. The significance of the compatible system function is that it is not a function of time and that several systems, <u>having the same compatible kernal</u>, may be combined by algebric means only.

In order to use the system function, it is necessary to show that:

$$y(\lambda) = x(\lambda)\ H(\lambda) \tag{2}$$

where: $Y(\lambda)$ is the λ-transform of $y(t)$

$X(\lambda)$ is the λ-transform of $x(t)$

$y(t)$ and $x(t)$ are related by the input/output

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* Laplace kernal $K_c(\lambda, t) = \exp(-st)$

** Frequently $K_c^{-1}(\lambda, t) = 1/K_c(\quad, t)$

and: $L_1(p,t)\ y(t) = L_2(p,t)\ x(t)$

The relation of equation (2) can be proved as follows [1]. First write $H(\lambda)$ in terms of the superposition integral:

$$H(\lambda) = \left[K_c^{-1}(\lambda,t)\right]^{-1} \int_{-\infty}^{\infty} h(t,\tau)\ K_c^{-1}(\lambda,\tau)\ d\tau \tag{3}$$

where $h(t,\tau)$ is the system response at time t to a unit impulse at the time τ. Next the output, y(t) can also be written as:

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau)\ x(\tau)\ d\tau \tag{4}$$

Substituting the general inverse transform for x(t) into equation (4) gives:

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau)\left\{\frac{1}{2\pi_j} \int_{C_\lambda} x(\lambda)\ K_c^{-1}(\lambda,t)\ d\lambda\right\} d\tau \tag{5}$$

where: C_λ is the contour of integration in the λ-domain. If the transform exists, the integral in equation (5) may be interchanged, yielding:

$$y(t) = \frac{1}{2\pi_j} \int_{C_\lambda} x(\lambda)\left[\int_{-\infty}^{\infty} h(t,\tau)\ K_c^{-1}(\lambda,\tau)\ d\tau\right] d\tau \tag{6}$$

Substituting equation (3) into equation (6) yields:

$$y(t) = \frac{1}{2\pi_j} \int_{C_\lambda} X(\lambda)\ H(\lambda)\ K_c^{-1}(\lambda,t)\ d\tau \tag{7}$$

The proof is completed by applying the direct generalized transform to equation (7) yielding:

$$y(\lambda) = X(\lambda)\ H(\lambda) \tag{2}$$

The system function is applied to determine compatible kernals in the next section.

E. Compatible Kernals. In this section determination of an inverse compatible kernal, $K_c^{-1}(\lambda,t)$ from the given input/output relation is considered. First the problem will be considered for first order $L_1(p,t)$.

Example 2-7

Given:

$$a_1(t)\ py(t) + a_o(t)\ y(t) = L_2(p,t)\ x(t)$$
$$a_1(t) \neq 0;\ t \geq 0 \tag{1}$$

Determine: a compatible kernal.

In this development [6] assume:

$$X(t) = K_c^{-1}(\lambda,t) \tag{2}$$

Then, by definition of the compatible system function:

$$y(t) = H(\lambda)\ K_c^{-1}(\lambda,t) \tag{3}$$

Substituting equations (2) and (3) into equation (1) yields:

$$\left[a_1(t)p + a_o(t)\right] H(\lambda)\ K_c^{-1}(\lambda,t) = L_2(p,t)\ K_c^{-1}(\lambda,t) \tag{4}$$

Equation (4) may be re-arranged to give:

$$a_1(t)p\ K_c^{-1}(\lambda,t) + \left[a_o(t) - \frac{L_2(p,t)}{H(\lambda)}\right] K_c^{-1}(\lambda,t) = 0 \tag{5}$$

and then:

$$p\ K_c^{-1}(\lambda,t) + \left[\frac{a_o(t)}{a_1(t)} - \frac{L_2(p,t)}{a_1(t)\ H(\lambda)}\right] K_c^{-1}(\lambda,t) = 0 \tag{6}$$

Since $L_1(p,t)$ is first order, $L_2(p,t)$ can be written as:

$$L_2(p,t) = b_1(t)p + b_o(t) \tag{7}$$

or, if $b_1(t) = 0$, then

$$L_2(p,t) = b_o(t) \tag{8}$$

It is also possible for $b_o(t)$ to be zero. The development for this case is similar to the first two cases and will not be considered. First consider the case where: $L_2(p,t) = b_o(t)$. Substituting $b_o(t)$ for $L_2(p,t)$ in equation (6) and solving for $K_c^{-1}(\lambda,t)$ gives:

$$K_c^{-1}(\lambda,t) = K_o \exp\left[-a(t) - \frac{b(t)}{H(\lambda)}\right] \tag{9}$$

where

$$a(t) = \int^t \frac{a_o(\tau)}{a_1(\tau)}\, d\tau \tag{10}$$

$$b(t) = \int^t \frac{b_o(\tau)}{a_1(\tau)}\, d\tau \tag{11}$$

It is of interest to note in the special case where $b(t) = bt$, where b is a complex constant, the result is similar to the integrating factor solution. Now, consider the second case where:

$$L_2(p,t) = b_1(t)p + b_o(t) \tag{12}$$

Substituting equation (12) into equation (4) and re-arranging terms yields:

$$\left[a_1(t)\ H(\lambda) + b_1(t)\right]p\ K_c^{-1}(\lambda,t) + \left[a_o(t)H(\lambda) - b_o(t)\right]K_c^{-1}(\lambda,t) = 0 \tag{13}$$

Solving equation (13) (by integrating factor method) yields:

$$K_c^{-1}(\lambda,t) = K_o \exp \left\{ \int^t \frac{H(\lambda)\ a_o(\tau) - b_o(\tau)}{H(\lambda)\ a_1(\tau) - b_1(\tau)}\, d\tau \right\} \tag{14}$$

provided that:

$$a_1(t)\ H(\lambda) - b_1(t) \neq 0\ ; \quad t \geq 0 \tag{15}$$

In the evaluation of equations (9) and (14) for $K_c^{-1}(\lambda,t)$, the selection of $H(\lambda)$ is not critical [6]. A convenient selection is:

$$H(\lambda) = \frac{1}{\lambda} \qquad \text{or} \qquad H(\lambda) = \lambda \tag{16}$$

The procedure of example 2-7 may be extended to higher order systems [6] as follows:

Example 2-8:

Given:

$$L_1(p,t)\ y(t) = L_2(,t)\ x(t) \tag{17}$$

Determine a compatible kernal.

As in example 2-7:

$$X(t) = K_c^{-1}(\lambda,t)$$

Substituting for y(t) and x(t) in equation (17) yields:

$$\left[H(\lambda)\ L_1(p,t) - L_2(p,t)\right] K_c^{-1}(\lambda,t) = 0 \tag{18}$$

The solution of equation (18) for $K_c^{-1}(\lambda,t)$ is generally no easier than the direct solution of the system differential equation. Once the inverse compatible kernal has been determined from equation (18), the compatible kernal* is generally given by:

$$K_c^{-1}(\lambda, t) = \frac{1}{K_c(\lambda, t)} \tag{19}$$

When both the direct and inverse kernals have been determined, it is necessary to return to the generalized transform pair and ensure that the integrals exist [6].

In addition to the difficulties of determinating compatible kernal and the existence problems associated with the corresponding generalized transform pair, two significant application problems arise. First, if the compatible kernal is to be applied to systems, the kernal must be generated either physically or on a computer. For example, assume that the kernal is of the form:

$$K_c(\lambda, t) = K_o \exp\left[-\lambda \sin t\right] \tag{20}$$

Such a kernal is not un-likely. But this kernal would be hard to generate. The second problem is visualizing the compatible kernal. Such a kernal as given in equation (20) is difficult, if not impossible to visualize. Additionally, even if a given compatible kernal could be generated and visualized it <u>will apply only</u> to the small class of

---

* The result is obvious for the Laplace transforms. Equation (19) holds whenever $K_c(\lambda, t)$ is a solution of the direct differential equation and $K_c^{-1}(\lambda, t)$ is a solution of the adjoint differential equation [10].

time varying systems for which it was developed. Even the addition of one order (e.g. if $L_1(t)$ becomes third order) will require a re-determination of the compatible kernal.

F. Incompatible System Function. The problem of generating and visualizing a compatible kernal can be avoided if the system function is allowed to be a function of time. In this case it becomes possible to select the kernal so that the results have a more familiar interpretation. Specifically, either the Laplace or the Fourier kernal can be selected. Either the Fourier selection:

$$H(j\omega, t) = \frac{y(t)}{e^{j\omega t}} \tag{1}$$

or the Laplace selection:

$$H(s,t) = \frac{y(t)}{e^{st}} \tag{2}$$

offers advantage because of the nature of simple exponential operations. The systems functions given in equation (1) and (2) are steady state expressions. The development, by Johnson and Kilmer [1], is similar to that presented in section D for the compatible system function. The following examples illustrate the procedure for determining $H(s,t)$.

Example 2-9:

Let the input/output relation:

$$L_1(p,t)\ y(t) = L_2(p,t)\ x(t) \tag{3}$$

be:

$$a_1(t)p\ y(t) + a_o(t)\ y(t) = x(t) \tag{3-a}$$

where:

$$x(t) = e^{st} \tag{3-b}$$

$$y(t) = H(s,t)\ e^{st} \tag{3-c}$$

Substituting the assumed values of the input and output into equation (3-a) yields:

$$a_1(t)p\ H(s,t)e^{st} + a_o(t)\ H(s,t)s^{st} = e^{st} \tag{4}$$

After taking the derivative of the first term, dividing both sides by exp(st) gives:

$$a_1(t)[p+s]\ H(s,t) + a_o(t)\ H(s,t) = 1 \tag{5}$$

Rearranging like derivative terms yields:

$$a_1(t)p\ H(s,t) + [a_o(t) + s\ a_1(s,t)]\ H(s,t) = 1 \tag{6}$$

Equation (6) may be solved by the integrating factor method for $H(s,t)$. The integrating factor, u, is:

$$u(t) = \exp\left[\int^t \frac{a_o(\tau) + s\ a_1(\tau)}{a_1(\tau)}\, d\tau\right] \tag{7}$$

Using the integrating factor, equation (7), the solution is:

$$H(s,t) = \exp\left[-\int^t \frac{a_o(\tau)+s\ a_1(\tau)}{a_1(\tau)}\, d\tau\right]\int_o^t \exp\left[\int^t \frac{a_o(\rho)+sa_1(\rho)}{a_1(\rho)}\, d\rho\right] d\tau \tag{8}$$

A comparison of equation (8) with the time invariant result is obtained if:

$$a_o(t) = A_o \tag{9}$$

$$a_1(t) = A_1 \tag{10}$$

Substituting equation (9) and (10) into equation (8) and evaluating the integrals yields:

$$H(s,t) = \exp\left[-\left(\frac{A_o}{A_1} + s\right)t\right] \frac{\frac{1}{A_1}\left[\exp\left[\left(\frac{A_o}{A_1} + s\right)t\right] - 1\right]}{\frac{A_o}{A_1} + s} \tag{11}$$

After expanding and cancelling terms:

$$H(s,t) = \frac{1}{A_o + A_1 s} - \frac{1}{A_o + A_1 s} \exp\left[-\left(\frac{A_o}{A_1} + s\right)t\right] \tag{12}$$

Using the Laplace transforms the H(s) obtained for the indicated condition is:

$$H(s) = \frac{1}{A_o + A_1 s} \tag{13}$$

Equations (13) and (12) are equal only in the steady state and only if the second term of equation (12) decays to zero in the steady state.

G. Combination of Incompatible System Functions. If H(s,t) is known for two systems, a natural problem is to determine the system function of a combination of the two systems.

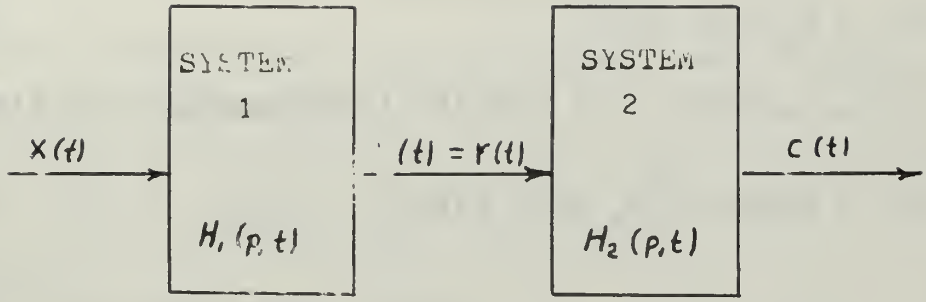
Example 2-9:

For the cascade combination indicated in figure 2-3, determine the combined system function, $H_t(s,t)$.

The solution of this problem, due to Zadeh [4], is as follows:

By the definition of the system sunction, $H_t(s,t)$

$$c(t) = H_t(s,t)e^{st} \quad \text{for } x(t) = e^{st} \tag{1}$$



System 1:

$$L_1(p,t)\,y(t) = L_2(p,t)\,x(t)$$

System 2:

$$L_3(p,t)\,c(t) = L_4(p,t)\,r(t)$$

Total System:

$$L_5(p,t)\,c(t) = L_6(p,t)\,x(t)$$

Transfer Functions:

$$H_1(p,t) = \frac{L_2(p,t)}{L_1(p,t)}$$

$$H_2(p,t) = \frac{L_4(p,t)}{L_3(p,t)}$$

Figure 2-3: <u>Figure for Example 2-9</u>

From the figure 2-3, it is possible to write

$$y(t) = H_1(p,t)\ x(t) \tag{2}$$

$$c(t) = H_2(p,t)\ y(t) \tag{3}$$

Substituting equation (2) and (3) into equation (1) yields

$$c(t) = H_2(p,t) \left[H_1(p,t)\ x(t) \right] \tag{4}$$

Since:

$$x(t) = e^{st} \tag{5}$$

Using the derivative properties of the exponential equation (4) becomes:

$$c(t) = H_2(p,t) \left[e^{st}\ H_1(s,t) \right] \tag{6}$$

and then:

$$c(t) = e^{st}\ H_2(p+s)\ H_1(s,t) \tag{7}$$

Substituting equation (1) into equation (7) yields:

$$e^{st}\ H_t(s,t) = e^{st}\ H_2(p+s,t)\ H_1(s,t) \tag{8}$$

Therefore:

$$H_t(s,t) = H_2(p+s,t)\ H_1(s,t) \tag{9}$$

The result, equation (9) can now be used to define some network functions.

H. <u>Basic Network Functions</u>. Several examples of network functions are first developed following the approach of Zadeh [4]. For a one port network with v(t) the voltage across the port and i(t) the current into the port. The input inpedance [4] is:

$$Z(s,t) \equiv \left.\frac{v(t)}{i(t)}\right|_{i(t) = e^{st}} \tag{1}$$

The input admittance [4] is:

$$Y(s,t) \equiv \left.\frac{i(t)}{v(t)}\right|_{v(t) = e^{st}} \tag{2}$$

The system differential equation is:

$$L_1(p,t)\ y(t) = L_2(p,t)\ i(t) \tag{3}$$

Substituting equation (1) into equation (3) yields:

$$L_1(p+s,t)\ Z(s,t) = L_2(s,t) \tag{4}$$

Substituting equation (2) into equation (3) yields:

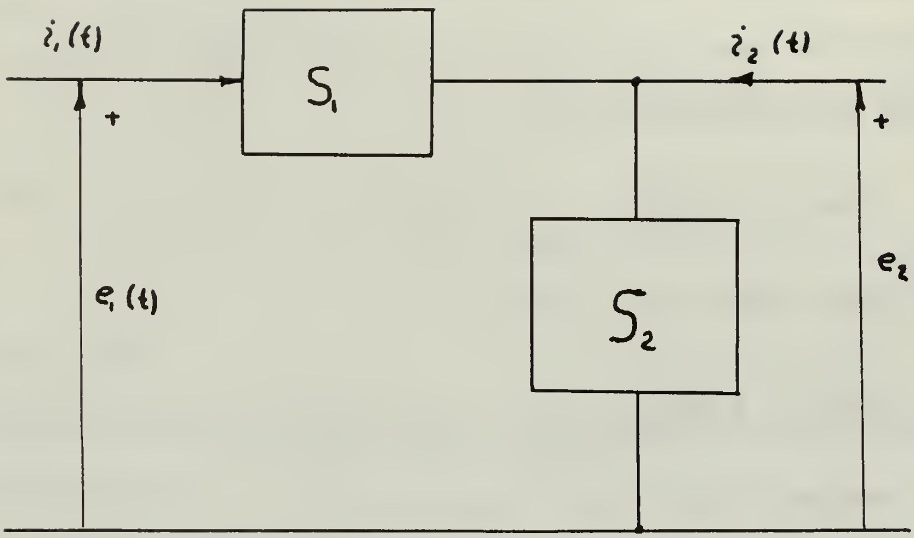
$$L_1(s,t) = L_2(p+s,t)\ Y(s,t) \tag{5}$$

For a two port network, the gain transfer function can be obtained in a similar manner [4]. For the network indicated in figure 2-4 Let:

$$G(s,t) \equiv \left.\frac{e_2(t)}{e_1(t)}\right|_{e_1(t) = e^{st}} \tag{6}$$

The network differential equation is:

$$L_1(p,t)\ e_2(t) = L_2(p,t)\ e_1(t) \tag{7}$$



$S_1 \triangleq$ System one = $Z_1(S,t)$

$S_2 \triangleq$ System two = $Z_2(S,t)$

For the voltages and currents:

subscript 1 $\triangleq$ input

subscript 2 $\triangleq$ output

Figure 2-4: Figure for Example 2-10

Substituting equation (6) into equation (7) yields:

$$L_1(p+s,t)\ G(s,t) = L_2(s,t) \tag{8}$$

For a given network, figure 2-4, the gain function, $G(s,t)$ or $G(j\omega,t)$ can be determined in a manner similar to the method used for time invariant systems provided the relation of equation 2-G-9 is observed.

Example 2-10:

For the network of figure 2-4, where system S_1 is $Z_1(s,t)$ and system S_2 is $Z_2(s,t)$, determine $G(s,t)$.

From figure 2-5:

$$i_1(t) = \left[Z_1(p,t) + Z_2(p,t)\right]^{-1} e_1(t) \tag{9}$$

and:

$$e_2(t) = Z_2(p,t)\ i_1(t) = Z_2(p,t)\left[Z_1(p,t)+Z_2(p,t)\right]^{-1} e_1(t) \tag{10}$$

Assuming:

$$e_1(t) = e^{st}$$

Solving equation (10) yields:

$$G(s,t) = Z_2(p+s,t)\left[Z_1(s,t)+Z_2(s,t)\right]^{-1} \tag{11}$$

If S_2 is assumed to be time invariant, as in the next example, then:

$$G(s,t) = Z_2(p+s)\ \left[Z_1(s,t)+Z_2(s,t)\right]^{-1} \tag{12}$$

Example 2-11:

Let:

$Z_2(s) = R$ be a fixed resistor

$Z_1(s,t) = L(t)s + \dot{L}(t)$ be a time varying inductor.

Determine the network differential equation starting from the network gain function of equation (12).

Substituting the assumed impedances into equation (12) yields:

$$G(s,t) = R\left[R + L(t)s + \dot{L}(t)\right] \qquad (13)$$

Substituting for G(s,t), the voltage ratio, yields:

$$RE_2(s,t) = \left[R + L(t)s + \dot{L}(t)\right] E_1(s,t) \qquad (14)$$

Writing equation (14) in the time domain yields:

$$Re_2(t) = Re_1(t) + L(t)\frac{d}{dt} e_1(t) + e_1(t)\frac{d}{dt} L(t) \qquad (15)$$

$$= Re_1(t) + \frac{d}{dt}\left(L(t)\ e_1(t)\right) \qquad (16)$$

$$e_2(t) = e_1(t) + \frac{1}{R}\frac{d}{dt}\left[L(t)\ e_1(t)\right] \qquad (17)$$

Equation (17) is the expected differential equation for the given system. Working backward, as in example 2-11, provides a measure of confidence in the development of the gain transfer function.

I. Conclusion. In contrast with the time invariant system transform method, the transform method for time varying systems is complicated by the fact that a single kernal is not applicable to all time varying systems. If the kernal is so chosen that the system function is not dependent on time (compatible), the kernal function and transform will generally be difficult to generate. Conversely, if the kernal is chosen for conceptual convenience the resulting system function for interconnected blocks will not generally be algebric.

Additionally, the transform method assumes that all component blocks have the same functional variation. This assumption is generally too restrictive.

All of the above reasons restrict the general usefulness of the transform approach to time varying systems.

3. STATE VARIABLE METHODS

A. Introduction. It is assumed that the basic methods of writing the state equations [18]:

$$x(t) = \underline{A}(t)\ x(t) + \underline{B}(t)\ u(t) \tag{1}$$

$$y(t) = \underline{C}(t)\ x(t) \tag{2}$$

where: $x(t)$ is the state vector

$\underline{A}(t)$ is the state matrix

$\underline{B}(t)$ is the system input matrix

$u(t)$ is the system input vector

$y(t)$ is the system output vector

$\underline{C}(t)$ is the linear transformation matrix between the state and output vectors.

are understood. The solution of equation (1) is given by:

$$x(t) = \underline{\phi}(t,t_o)\ x(t_o) + \int_{t_o}^{t} \underline{\phi}(t+\tau)\ \underline{B}(\tau)\ u(\tau)\ d\tau \tag{3}$$

where: $\underline{\phi}(t,t_o)$ is the state transition matrix for time t, and initial time t_o.

The solution depends upon determining either an explicit or tabular expression for the state transition matrix. In this chapter several methods of determining or approximating the state transition matrix are considered.

B. Exponential Form of the State Transition Matrix. The state variable approach for time varying systems is similar to the approach used in time invariant systems with two important exceptions. The exceptions are noted by Zadeh and Desoer [7]. The following discussion is intended to relate these exceptions to the context of this thesis. The first exception concerns the use of the exponential form of the state transition matrix. The second exception concerns the effect of an initial time other than zero on the state transition matrix.

The first exception concerns the exponential expansion for the state transition matrix:

$$\underline{\phi}(t,t_o) = \exp\left[\int_{t_o}^{t} \underline{A}(\tau)\,d\tau\right] \tag{1}$$

Equation (1) is valid if and only if:

$$\underline{A}(t) \int_{t_o}^{t} \underline{A}(\tau)\,d\tau = \left[\int_{t_o}^{t} \underline{A}(\tau)\,d\tau\right] A(t) \tag{2}$$

Equation (2) is the commutation requirement* on $\underline{A}(t)$ and the integral of $\underline{A}(t)$. The necessity for this requirement can be seen from the defining equations:

$$\dot{x}(t) = \underline{A}(t)\; x(t) \;; \quad x(t_o) = x \;. \tag{3}$$

$$\dot{\underline{\phi}}(t,t_o) = \underline{A}(t)\; \underline{\phi}(t,t_o) \;; \quad \phi(t_o,t_o) = \underline{I} \tag{4}$$

---

* The state transition matrix can be expressed as an exponential function if and only if the integral commutation requirement is satisfied [7]. For time invariant linear systems the state matrix will always commute with its integral.

If the exponential expansion (1) is substituted into equation (4) then:

$$\underline{\phi}(t,t_o) = \exp\left[\int_{t_o}^{t} \underline{A}(\tau)\,d\tau\right] = I + \int_{t_o}^{t} \underline{A}(\tau)\,d\tau + \frac{1}{2!}\left[\int_{t_o}^{t} A(\tau)\,d\tau\right]^2 + \cdots \tag{5}$$

The exponential matrix is expanded as:

$$\underline{\phi}(t,t_o) = I + \int_{t_o}^{t} \underline{A}(\tau)\,d\tau + \frac{1}{2!}\left[\int_{t_o}^{t} \underline{A}(\tau)\,d\tau\right]^2 + \frac{1}{3!}\left[\int_{t_o}^{t} \underline{A}(\tau)\,d\tau\right]^3 + \cdots \tag{6}$$

The derivative of the exponential matrix, equation (6), is:

$$\frac{d}{dt}\left[\underline{\phi}(t,t_o)\right] = A(t) + \left[\int_{t_o}^{t} \underline{A}(\tau)\,d\tau\right]A(t) + \frac{1}{2!}\left[\int_{t_o}^{t} \underline{A}(\tau)\,d\tau\right]^2 + \cdots \tag{7}$$

$$\frac{d}{dt}\left[\underline{\phi}(t,t_o)\right] = \exp\left[\int_{t_o}^{t} \underline{A}(\tau)\,d\tau\right] \underline{A}(t) \tag{8}$$

Now, if the exponential expansion, equation (6) is to be valid, equation (8) and equation (5) must be equal. The required equality will hold if and only if the commutation on $\underline{A}(t)$, equation (2), holds. If the commutation requirement holds, the $\underline{A}(t)$ may be extracted from equation (7) or equation (8) as a pre-multiplication term. If this extraction is possible, then equation (8) and equation (5) are equal.

The second exception is related to the group property [7] of the state transition matrix.

$$\underline{\phi}(t_1,t_2)\,\underline{\phi}(t_2,t_3) = \underline{\phi}(t_1,t_3) \quad \text{for all } t_1,\ t_2,\ t_3$$

$$\underline{\phi}^{-1}(t_1,t_2) = \underline{\phi}(t_2,t_1) \quad \text{for all } t_1,\ t_2 \tag{9}$$

For the specific times: $t_1 = 0$, $t_2 = t_o$ (initial time), and $t_3 = t$ (observation time), property (9) gives:

$$
\begin{aligned}
\underline{\phi}(0,t_o)\ \underline{\phi}(t_o,t) &= \underline{\phi}(0,t) \\
\underline{\phi}(t_o,t) &= \underline{\phi}^{-1}(0,t_o)\ \underline{\phi}(0,t) \\
\underline{\phi}(t,t_o) &= \left[\underline{\phi}^{-1}(0,t_o)\ \underline{\phi}(0,t)\right]^{-1} \\
\underline{\phi}(t,t_o) &= \underline{\phi}(t,0)\ \underline{\phi}^{-1}(t_o,0)
\end{aligned}
\tag{10}
$$

If the system is time invariant, then:

$$\underline{\phi}(t,t_o) = \underline{\phi}(t-t_o) \tag{11}$$

and the transition matrix is a linear function of the time difference. Time varying systems do not generally have this property. Specifically, consider a system for which the commutation requirement on $\underline{A}(t)$ is satisfied. Then:

$$
\begin{aligned}
\underline{\phi}(t,t_o) &= \exp\left[\int_0^t \underline{A}(\tau)\,d\tau + \int_{t_o}^0 \underline{A}(\tau)\,d\tau\right] \\
&= \exp\left[\underline{B}(t)-\underline{B}(0)+\underline{B}(0)-\underline{B}(t_o)\right] = \exp\left[\underline{B}(t)-\underline{B}(t_o)\right]
\end{aligned}
\tag{12}
$$

where:

$$\underline{B}(t) \triangleq \int^t A(\tau)\,d\tau$$

Unless $\underline{B}(t)$ is a linear function of the time difference, the state transition matrix will not be a linear function of the time difference.

C. <u>Determination of the State Transition Matrix</u>. If the state matrix, $\underline{A}(t)$, is in the phase variable form:

$$\dot{x}(t) = \left[\begin{array}{c:c} 0 & I \\ \hdashline \multicolumn{2}{c}{a^T} \end{array}\right] x(t) \qquad (1)$$

$$a^T = \left[a_o(t),\ a_1(t)\ \cdots\ a_{n-1}(t)\right]$$

Then the state transition matrix is:

$$\underline{\phi}(t,t_o) = \begin{bmatrix} \phi_{11}(t,t_o) \cdots\cdots\cdots & \phi_{1n}(t,t_o) \\ \frac{d}{dt}\ \phi_{11}(t,t_o) \cdots\cdots & \frac{d}{dt}\ \phi_{1n}(t,t_o) \\ \vdots & \vdots \\ \frac{d^{n-1}}{dt^{n-1}}\ \phi_{11}(t,t_o) \cdots & \frac{d^{n-1}}{dt^{n-1}}\ \phi_{1n}(t,t_o) \end{bmatrix} \qquad (2)$$

It is possible to use the group property:

$$\underline{\phi}(t,t_o) = \underline{\phi}(t,0)\ \phi^{-1}(t_o,0) \qquad (3)$$

of the state transition matrix to establish a procedure for determining the state transition matrix. The basis for the procedure is the determination* of $\underline{\phi}(t,0)$. The first column of $\underline{\phi}(t,t_o)$ is the response of the undriven system to the set of initial conditions:

$$x(0) = \left[1,\ 0,\ \cdots\ 0\right]^T \qquad (4)$$

---

* This procedure is presented for the phase variable form, but is not limited to this case only.

The other columns may be determined by varying the state having the unit initial condition. Consider the second order system shown in the following example:

Example 3-1:

$$\dot{x}(t) = \underline{A}(t)\ x(t) = \begin{bmatrix} 0 & 1 \\ \cos t & \sin t \end{bmatrix} x(t) \qquad (5)$$

The state matrix, $\underline{A}(t)$, for this system* is periodic but does not satisfy the commutation requirement. It is possible to determine the response of the system by a Runge-Kutta technique for the two sets of unit initial state:

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (6)$$

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (7)$$

Some of the results of the calculation are given in table I. The data from table I can now be used to define the state transition matrix for a given observation time and initial time. The example is now continued for the initial condition:

$$x(t_o) = x(1\ \text{second}) = \begin{bmatrix} 15 \\ 5 \end{bmatrix} \qquad (8)$$

---

* It is of interest to note that this particular system is not asymptotically stable in the large.

TABLE I

Selected Values of the State Transition Matrix, $\phi(t,0)$, for Example 3-1

Time	$\phi_{11}(t,0)$	$\phi_{21}(t,0)$	$\phi_{12}(t,0)$	$\phi_{22}(t,0)$
0.0	1.0	0.0	0.0	1.0
0.4	1.0821	4.2140	4.2167	1.1642
0.8	1.3543	9.7151	9.8108	1.7037
1.0	1.5835	1.3325	1.3641	2.1478
1.4	2.2933	2.2599	2.4619	3.4261
1.8	3.4115	3.3224	4.1533	5.0447
2.0	4.1210	3.7474	5.2370	5.7621
2.4	5.6824	3.8386	7.6891	6.1941
2.8	6.9742	2.3369	9.8761	4.3092
3.0	7.3156	1.0331	1.0564	2.4917

Note: The values in this table are a result of a forth order Runge-Kutta solution of the state equations (see Appendix I).

and observation time of two seconds. First it is necessary to write the state transition matrix at the initial time.

$$\underline{\phi}(t_o,0) = \underline{\phi}(1,0) = \begin{bmatrix} 1.5835 & 1.3641 \\ 1.3325 & 2.1470 \end{bmatrix} \tag{9}$$

Next, from table I, write the state transition matrix at the observation time: t:

$$\underline{\phi}(t,0) = \underline{\phi}(2,0) = \begin{bmatrix} 4.1210 & 5.2370 \\ 3.7474 & 5.7621 \end{bmatrix} \tag{10}$$

Using the group property of the state transition matrix, the state transition matrix between the initial time and the observation time may be written as the product of equation (10) and the matrix inverse of equation (9).

$$\underline{\phi}(t,t_o) = \underline{\phi}(t,0)\underline{\phi}^{-1}(t_o,0) = \underline{\phi}(2,0)\,\phi^{-1}(1,0) = \begin{bmatrix} 1.18277 & 1.68711 \\ 0.23410 & 2.53411 \end{bmatrix} \tag{11}$$

With the desired state transition matrix known, the response at the observation time is:

$$x(t) = x(2 \text{ seconds}) = \phi(2,1)x(1) = \begin{bmatrix} 26.117 \\ 16.182 \end{bmatrix} \tag{12}$$

A Runge-Kutta solution for the same initial conditions and period gives:

$$x(2 \text{ seconds}) = \begin{bmatrix} 26.177 \\ \\ 16.188 \end{bmatrix} \tag{13}$$

The preceeding procedure is useful when the response at a fixed observation time is desired for a number of initial conditions and initial times. Comparisons of the state vectors, equations (12) and (13), indicates that the problem of computational accuracy must be considered.

Another method, of determining the state transition matrix, is to assume that the state matrix, $\underline{A}(t)$, is constant over a small interval of time, T. The transition property of the state transition matrix permits the following expression for the state vector at the end of each interval*:

$$x\left[(K+1)T\right] = \underline{\phi}\left(\underline{A}(KT), T\right) x(KT) \tag{14}$$

Where: $\underline{\phi}(\underline{A}(KT),T)$ represents the time invariant state transition matrix during the interval.

---

* In developing the conditions for a series solution for the state transition matrix, Hahn [10] has used a similar approach. However, Hahn's presentation is oriented toward proving the integral commutation requirement on $\underline{A}(t)$ rather than toward obtaining a numerical approximation to the state response.

Specifically, for the first interval:

$$\underline{\phi}(T,0) = \underline{\phi}\left[\underline{A}(0),T\right] = \exp\left[\underline{A}(0)T\right] \tag{15}$$

where: $\underline{A}(0)$ is the approximate value of $\underline{A}(t)$ for the first interval.

$\underline{A}(KT)$ is the approximate value of $\underline{A}(t)$ for: $KT \le t \le (K+1)T$.

In evaluating equation (14), several approximations of $\underline{A}(KT)$ are possible. Two obvious approximations are to let $\underline{A}(KT)$ equal the value of $\underline{A}(t)$ at the beginning of the interval or to let $\underline{A}(KT)$ equal the average value of $\underline{A}(t)$ during the interval. It is also possible to compute the state transition matrix, $\underline{\phi}[\underline{A}(KT),T]$, either as an exponential series for each interval or as the approximation:

$$\underline{\phi}\left[\underline{A}(KT),T\right] = \underline{\phi}\left[\underline{A}\left((K-1)T\right),T\right]\left[\underline{I} + T\,\Delta\underline{A}(KT)\right] \tag{16}$$

$$\Delta A(KT) = \underline{A}(KT) - A\left[(K-1)T\right] \tag{17}$$

Equation (16) results from neglecting the higher order terms from the series expansion of:

$$\exp\left[[\underline{A}(K-1)T]T + \Delta\underline{A}(KT)T\right] \tag{18}$$

Equation (16) is valid only if the change in $\underline{A}(t)$ during the time interval, T, is small.

It is of interest to note that it is not possible to evaluate equation (18) as a product of two exponentials for all $\underline{A}((K-1)T)$ and $\underline{A}(KT)$. In general [19]:

$$\exp\left[\underline{A}+\underline{B}\right] = \exp\left[\underline{A}\right]\exp\left[\underline{B}\right] \tag{19}$$

if and only if $\underline{AB}$ equals $\underline{BA}$.

When the above procedure was applied to the system of example 3-1, the results deviated from the results given in example 3-1. In general it appears that the numerical errors resulting from the increased number of matrix multiplications required when the time interval, T, is sufficiently small (for near constant $\underline{A}(t)$ over the time interval) may limit the usefulness of this approach. Further research is required in this area.

4. REDUCTION TO TIME INVARIANT SYSTEMS

A. Introduction. Certain time varying systems are equivalent to time invariant systems when the coordinate basis of time invariant system is properly chosen[7]. The problem is to determine the proper coordinate basis for the time invariant system and the linear transform from the coordinate basis of the time varying system to the coordinate basis of the time invariant system.

B. Linear Transformation Matrix. In the solution of

$$\dot{x}(t) = \underline{A}(t)x(t) \tag{1}$$

x(t) is determined with respect to some fixed orthogonal basis. If a new time invariant system

$$\dot{y}(t) = \underline{C}y(t) \tag{2}$$

is to be considered equivalent to equation (1), then there must exist a linear transformation matrix [7] such that

$$x(t) = \underline{Q}(t)y(t) \tag{3}$$

Zadeh and Doeser [7] have discussed a transformation matrix when the linear transformation matrix, $\underline{Q}(t)$, has the properties of a Lyapunov transform. For a matrix to be a Lyapunov transform, the matrix must have the following properties [7]:

a. The matrix and its derivative are bounded on the closed interval (t_o, infinity).

b. The magnitude of the determinate of the transform must be greater than zero for all time greater than the initial time and less than time equal to infinity. The state transition matrix for a system whose states are bounded as time increases is an example of a Lyapunov transform.

It is not required that the equivalent time invariant system be unique. However, the set of Lyapunov transforms for a given system do form a group [7]. Hence the product of any two transforms or the product of a transform and a constant matrix will also be a Lyapunov transform of the same set. This group property will be used later to develop a relation between $\underline{Q}(t)$ and the state transition matrix for some systems. Before this subject is considered, it is desirable to indicate the procedure for determining $\underline{Q}(t)$.

Following Zadeh and Doeser [7], $\underline{Q}(t)$ may be developed from equations (1), (2), (3) as follows:

First differentiate equation (3)

$$\dot{x}(t) = \dot{\underline{Q}}(t)y(t) + \underline{Q}(t)\dot{y}(t) \tag{4}$$

then substitute equation (1)

$$\underline{A}(t)x(t) = \dot{\underline{Q}}(t)y(t) + Q(t)\dot{y}(t) \tag{5}$$

and equation (3)

$$\underline{A}(t)\underline{Q}(t)y(t) = \dot{\underline{Q}}(t)y(t) + \underline{Q}(t)\dot{y}(t) \tag{6}$$

solving equation (6) for $\dot{y}(t)$ gives

$$\dot{y}(t) = Q^{-1}(t) \left\{\underline{A}(t)\underline{Q}(t) - \dot{\underline{Q}}(t)\right\}y(t) \tag{7}$$

The inverse of $\underline{Q}(t)$ will always exist if $\underline{Q}(t)$ exists since the determinate of $\underline{Q}(t)$ was required to be non zero (see definition of Lyapunov transforms above). Now comparing equations (7) and (2) gives:

$$\underline{C} = \underline{Q}^{-1}(t) \left\{\underline{A}(t)Q(t) - \dot{Q}(t)\right\} \tag{8}$$

The constant matrix $\underline{C}$ may be varied to generate a set of equivalent time invariant systems. The matrix $\underline{C}$ may generally be selected for convenience of solution. For an arbitrary system, a suitable $\underline{C}$ matrix may not exist. However, if the state matrix, $\underline{A}(t)$, is periodic a $\underline{C}$ may always be found [7]. Hence, all periodic time varying systems are equivalent to some time invariant system. If it is possible to determine a $\underline{Q}(t)$ for the special case where the $\underline{C}$ matrix is the zero matrix, then the time varying system will be asymptotically stable in the large (ASIL)[7]*.

C. Relations Between Q(t) and the State Transition Matrix. The state transition matrix and its derivative

$$\dot{\underline{\phi}}(t,t_o) = \underline{A}(t)\underline{\phi}(t,t_o) \tag{1}$$

are continuous if $\underline{A}(t)$ is continuous and the states have been chosen such that the state must be physically continuous (i.e. voltage across capacitors). The magnitude of the determinate of the state transition matrix is given by [7]

$$|\det \underline{\phi}(t,t_o)| = \left| \exp \left\{\int_{t_o}^{t} \mathrm{tr}\, \underline{A}(\tau)\, d\tau\right\}\right| \tag{2}$$

---

* The state vector will approach zero as time approaches infinity.

where: tr $\underline{A}(t) \triangleq$ trace of $A(t)$ = Sum of diagonal elements of $\underline{A}(t)$.

Hence the second condition for a Lyapunov transform

$$|\det \underline{\phi}(t,t_o)| > \text{constant matrix} > \underline{0} \tag{3}$$

can be determined from $\underline{A}(t)$. Thus the state transition matrix, if it is continuous and satisfies equation (3), is a Lyapunov transform. Using the group property of the Lyapunov transform it is possible to write:

$$\underline{\phi}(t,t_o) = \underline{Q}(t)\ \underline{K} = \underline{K}_1\underline{Q}(t) \quad \text{for all } t_o \tag{4}$$

for well behaved state transition matrices and some constant matrix $\underline{K}$ or $\underline{K}_1$.

If the state vector x(t) for the system described by

$$\dot{x}(t) = \underline{A}(t)x(t) \tag{5}$$

approaches zero as time approaches infinity for all sets of initial conditions, then the system is asymptotically stable in the large (ASIL). Two necessary and sufficient conditions for ASIL [7] on the norm* of the state transition matrix are:

$$\|\underline{\phi}(t,t_o)\| \leq \text{constant matrix for all} \quad t \geq t_o \tag{6}$$

and

$$\lim_{t\to\infty} \|\underline{\phi}(t,t_o)\| = 0 \quad \text{for all } t_o \tag{7}$$

---

* $\|\underline{\phi}(t,t_o)\| \triangleq$ norm of $\underline{\phi}(t,t_o) = \phi^T(t,t_o)\ \underline{\phi}(t,t_o)$ in this Thesis.

By comparing equations (6) and (7) with equation (4), the ASIL conditions may be re-stated as:

$$\|\underline{Q}(t)\| \leq \text{constant matrix for all } t \geq t_o \tag{8}$$

$$K \leq \text{finite constant matrix} \tag{9}$$

and

$$\lim_{t \to \infty} \|Q(t)\| = 0 \quad \text{for all } t_o \tag{10}$$

Hence, the ASIL property of a time varying system can be determined from the bahavior of the norm of the linear transform to an equivalent time invariant system provided that the time invariant system has bounded states for all time greater than the initial time.

The limitation of this approach is that the determination of $\underline{Q}(t)$ from

$$\underline{C} = \underline{Q}^{-1}(t)\left[\underline{A}(t)\underline{Q}(t) - \dot{Q}(t)\right] \tag{8}$$

is in general no easier than the direct determination of the state transition matrix. This same problem was encountered in the general transform approach where it was noted that the determination of the transform solution is generally as difficult as the direct solution of the differential equation.

5. LINEAR FEEDBACK SYSTEMS

A. Introduction. In the previous chapters, methods of solving time varying systems were discussed. Other major areas of concern are stability, compensation, phase margin, and gain margin. In Chapters 2 and 3 it was seen that the solution of time varying systems is of orders of magnitude more difficult than the solution of a corresponding time invariant system, and that frequency domain methods do not generally simplify the solution. In this chapter, the emphasis is on linear time varying feedback systems and in particular the stability of such systems. In the later sections of the chapter, a special system is considered where the time variation is limited to the feedback loop.

B. Stability in the Large and Initial Conditions. Since the systems under consideration have been limited to linear time varying systems*, stability in the large for all sets of initial conditions can be established by considering the response of **

$$\dot{x}(t) = \underline{A}(t)x(t) \tag{1}$$

$$y(t) = \underline{C}x(t) \tag{2}$$

---

* In which all states can be reached from the initial state for any initial state [7].

** See Chapter 3.

to a set of independent initial conditions equal to the order, n, of the system. The stability being established can either be asymptotic stability in the large (ASIL) for which:***

$$\lim_{t\to\infty} \|x(t)\| = 0 \tag{3}$$

or stability in the large (SIL) for which:

$$\lim_{t\to\infty} \|x(t)\| \leq \text{finite constant} \tag{4}$$

Theorem 5-1: A linear time varying system will be stable in the large if and only is the system state vector, x(t), is bounded for each of the following initial conditions:

$$x(t_o) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots\cdots \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Theorem 5-2: A linear time varying system will be asymptotically stable in the large if and only if the system state vector, x(t), goes to zero for large time for each of the following initial conditions:

$$x(t) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

*** Norm $x(t) = \|x(t)\| \equiv x^T(t)x(t)$

These theorems will be established by first showing that the behavior of the norm is determined by the behavior of the state transition matrix. When this relation has been established, the theorem follows by showing that the columns of the state transition matrix are determined by the corresponding initial condition. That is, the i^{th} column of the state transition matrix is determined by the initial condition where the initial value of the i^{th} state is unity and the initial values of all other states are zero.

The first part of the proof is established by expanding the norm of the state vector. Since:

$$x(t) = \phi(t,t_o)x(t_o) \tag{5}$$

the state vector norm may be re-written as:

$$\|x(t)\| = x^T(t)x(t) = x^T(t_o)\phi^T(t,t_o)\phi(t,t_o)x(t_o) \tag{6}$$

Since the initial state vector, $x(t_o)$, has been assumed finite, the stability conditions of equations (3) and (4) may be re-written as:

$$\lim_{t\to\infty}\|\phi(t,t_o)\| = 0 \tag{7}$$

for ASIL and

$$\lim_{t\to\infty}\|\phi(t,t_o)\| \leq \text{finite constant matrix} \tag{8}$$

for SIL.

The second part of the proof may be established by observing the correspondence between the state vector for each of the initial conditions and the columns of the state transition matrix. Since:

$$x(t) = \begin{bmatrix} \phi_{11}(t,t_o) & \cdots & \phi_{1n}(t,t_o) \\ \vdots & & \vdots \\ \phi\ \ (t,t_o) & \cdots & \phi_{nn}(t,t_o) \end{bmatrix} x(t_o) \tag{9}$$

the state vector for the initial condition where the initial value of the i^{th} state is unity and the initial values for all other states are zero is:

$$x(t) = \begin{bmatrix} \phi_{1i}(t,t_o) \\ \vdots \\ \phi_{ni}(t,t_o) \end{bmatrix} \tag{10}$$

Equation (10) is just the i^{th} column of the state transition matrix. Now, using the notation: $x_i(t,t_o)$ to denote the response, $x(t)$, when the i^{th} state has the initial value, (at initial time, t_o) of unity and all other states have the initial value of zero, the state transition matrix may be written as:

$$\phi(t,t_o) = \left[x_1(t,t_o),\ x_2(t,t_o) \cdots\cdots x_n(t,t_o)\right] \tag{11}$$

Now, the norm of the state transition matrix becomes:

$$x^T(t,t_o)\phi(t,t_o) = \left[x_1(t,t_o)\cdots x_n(t,t_o)\right]^T\left[x_1\ldots x_n\right]$$

$$= \begin{bmatrix} x_1^T(t,t_o)x_1(t,t_o), & \cdots & x_1^T(t,t_o)x_n \\ \vdots & x_2^T(t,t_o)x_2(t,t_o) & \vdots \\ & \ddots & \\ x_n^T(t,t_o)x_1(t,t_o) & \cdots & x_n^T(t,t_o)x_n(t,t_o) \end{bmatrix}$$

If each of the responses, $x_i(t,t_o)$, is bounded, the matrix of equation (12) must also be bounded. Hence the system is stable in the large. Additionally, since the diagonal terms are sums of squares, the matrix will not go to zero* unless the responses, $x_i(t,t_o)$, all go to zero. Thus asymptotic stability in the large depends upon each of the n-responses going to zero.

Since the response to the finite set of initial conditions described above can be determined by a computer pro-

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* Asymptotic stability is slightly misleading when applied to time varying systems since the state transition matrix need not go to zero exponentially.

cedure, the stability of a given system can be determined. Combining this result with the similar result given in chapter 3 for determining the state transition matrix; it is possible to specify both the response and stability of a given system from the same computer analysis. The limitation of this method is that no provision has been made for any system compensation specification.

C. Time Varying Feedback. In this section the system with time variation restricted to the feedback loop is defined*(figure 5-1). This configuration is then discussed in more detail in the following sections. The time varying feedback is assumed to be either:

a. a pure linear gain,

b. a pure linear phase shift,

c. a combination of linear phase shift and linear gain.

The focus of the discussion will be on the effect of the values of the feedback component on system stability and system response when the input is zero. Results given are based on either a fourth order Runge-Kutta solution of the state equations or a state transition matrix difference equation solution. The details of the solution methods are given in Appendix I.

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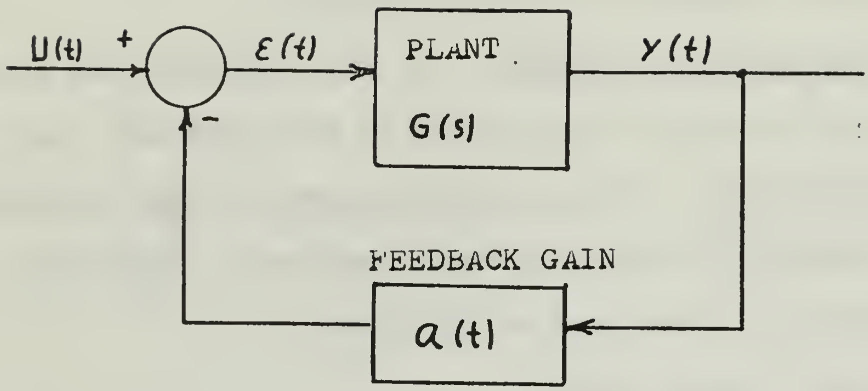
* The configuration of figure 5-1 has received considerable attention recently by Brockett [11,12].



$$G(s) = \frac{Q(s)}{P(s)}$$

Where: u(t) is the system input

ε(t) is the error function

G(s) is a fixed parameter plant

y(t) is the system output

a(t) is a linear time varying gain

Q(s) is the numerator polynominal of G(s)

P(s) is the denominator polynominal of G(s)

Figure 5-1: <u>Example of a Simple Time Varying Feedback System</u>

D. Nyquist Plot.** Since the plant is assumed fixed, it is possible to obtain a Nyquist diagram for the given G(s). Such a diagram is shown in Figure 5-2. The standard procedure for determining the Nyquist diagram are given in a number of texts [13,14]. In this section, consideration of the Nyquist diagram is limited to showing the effect of a fixed pure gain and a fixed pure phase shift on the Nyquist path. Specifically a fixed phase shift will rotate the path while a fixed positive gain will change the magnitude of the path.

In the case of a fixed gain, A_o, the resulting loop gain is $A_oG(j\omega)$. Now:

$$A_oG(j\omega) = A_o \, \mathrm{Re}G(j\omega) + A_o \, \mathrm{Im}\, G(j\omega) \quad A_o \neq 0 \tag{1}$$

Since both the real and imaginary parts of $G(j\omega)$ are multiplied by the same positive number, the resulting path is an enlargement of the original path.

In the case of a fixed phase shift, θ, the result is seen by writing $G(j\omega)$ in the polar form as:

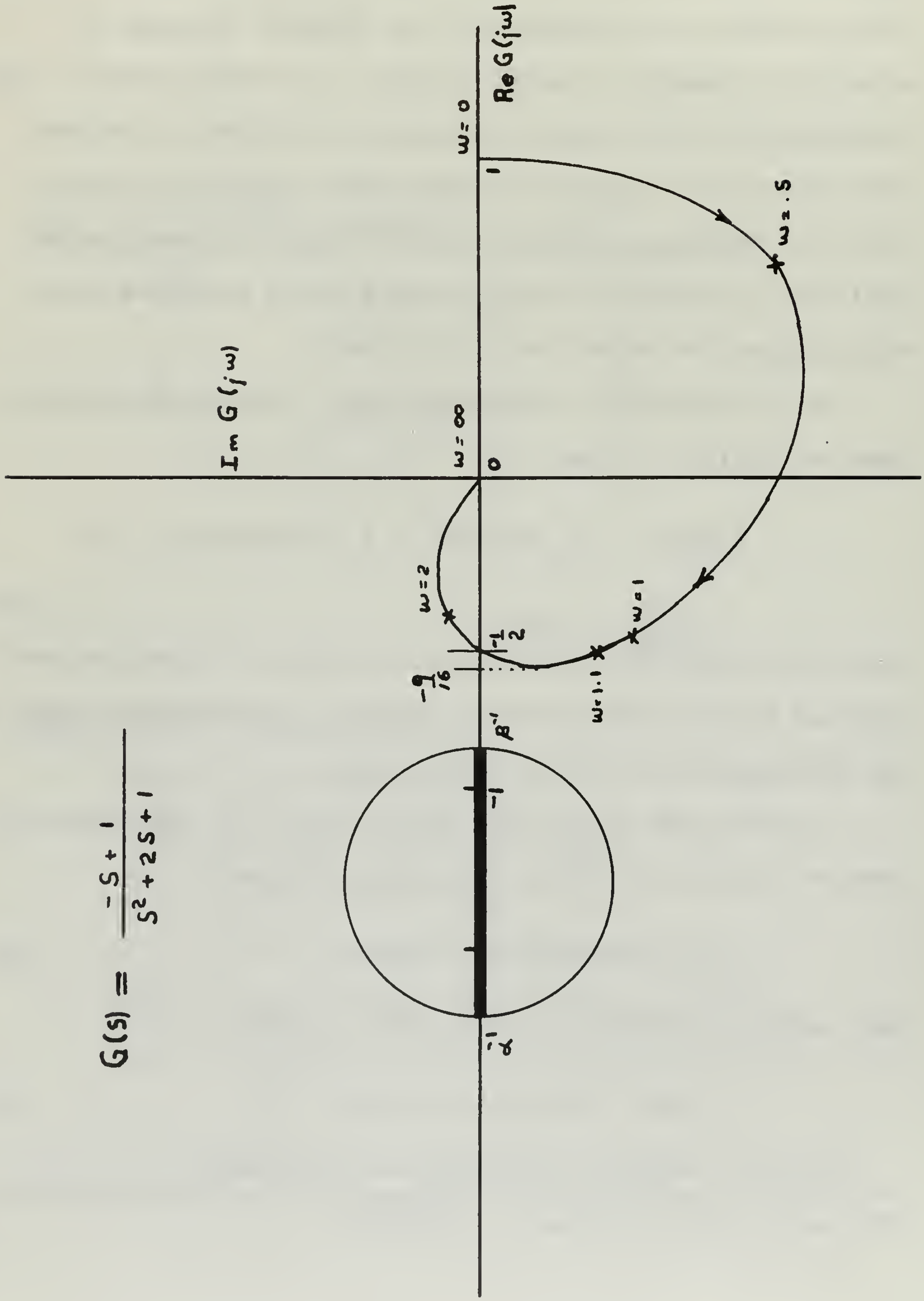
$$G(j\omega) = |G(j\omega)| \quad \angle G(j\omega) \tag{2}$$

Now, with the shift, the new path, $G_1(j\omega)$ is:

$$G_1(j\omega) = |G(j\omega)| \{\angle G(j\omega) + \theta\} \tag{3}$$

** Polar plot of $G(j\omega)$

Figure 5-2: <u>Example of a Nyquist Diagram</u>



For combinations of fixed gain increase and a fixed phase shift, the Nyquist path is both enlarged and rotated.

E. Circle Criteria for Pure Gain Feedback. The maximum and minimum values of the feedback gain for which the system is stable are determined by using the Nyquist diagram to study the stability of a time invariant system with a fixed gain feedback. Specifically, the Nyquist criteria states [11]:

If $G(s)$* has a number of poles, ρ, in the half plane, $Re(s) \geq 0$, then the system is stable if and only if the Nyquist path of $G(j\omega)$ makes ρ counterclockwise encirclements of the critical point equal to $-1/A$, where A is the fixed feedback gain.

If the feedback gain, A, is allowed to assume a series of fixed values, a minimum stable gain, α, and a maximum stable gain, β, can be determined. The allowable fixed gain reciprocals define a section of the negative $Re(j\omega)$ axis between β^{-1} and α^{-1}.

---

* Note: $G(s)$ is for the forward plant only. The feedback gain enters the criteria by shifting the critical point from minus one to the reciprocal of the negative of the feedback gain [11]. This version of the Nyquist criteria is more convenient for the presentation to follow.

This line* is shown on figure 5-2.

The circle criteria generalizes the Nyquist criteria to time varying systems**. The g-plane, with the critical circle indicated, is shown on figure 5-3. With a significant difference, the circle criteria merely replaces the critical point of the Nyquist criteria with a critical disc. The significant difference is that the criterion no longer is both a necessary and sufficient condition for stability. The circle criteria gives a sufficient condition for stability. The circle criteria has been discussed in several recent papers by Brockett [11,12,15].

If the feedback gain functions; a(t); is known, the circle criteria reduces to the Nyquist criteria point as the magnitude of the time variation becomes smaller.

In the circle criteria nothing has been stated about the shape or frequency components of the feedback gain. Since the criteria is sufficient for asymptotic stability, any functional form is allowed provided the maximum and minimum values are between the indicated limits.

In establishing necessary conditions for stability, it is necessary to consider only functions whose maximum and/or

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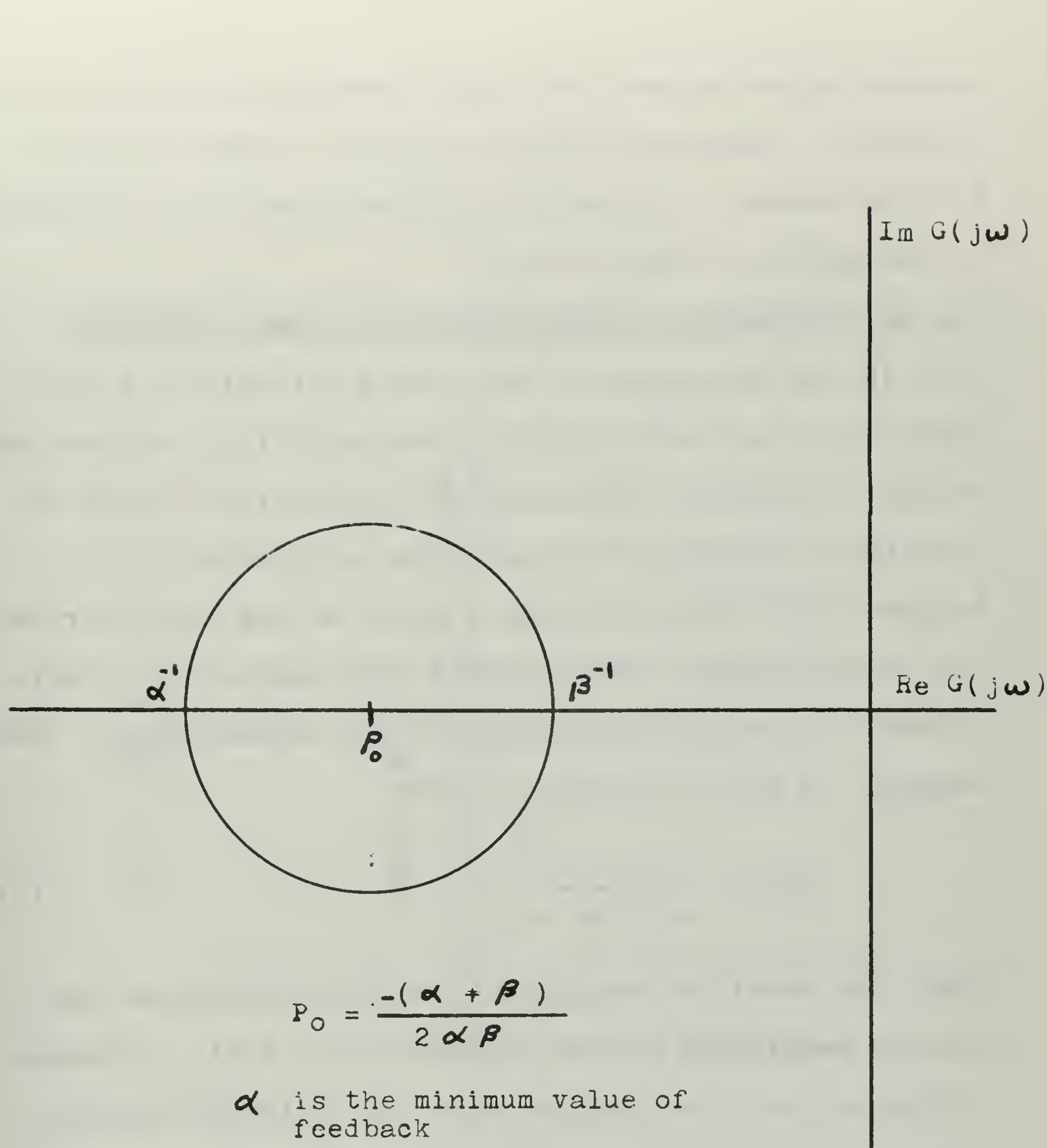
* For some systems it is possible that the line of stable fixed gains may be broken. In this thesis it is assumed that the line is continuous.

** The circle criteria is not limited to linear systems.



$$P_o = \frac{-(\alpha + \beta)}{2\alpha\beta}$$

α is the minimum value of feedback

β is the maximum value of feedback

Figure 5-3: <u>Critical Disc Shown on g-Plane</u>

minimum values exceed the limits determined by the circle criteria. Compensation may be added to ensure stability, but the concept of phase and gain margins [13,14] depends on the specific shape of a(t).

F. Instability Criteria for Pure Gain Feedback. In view of the limitation of the circle criteria to a sufficient condition for stability, Brockett [11], has developed some sufficient conditions for instability (based on the circle criteria) for the system of figure 5-1. Brockett [11] first develops a proof of the circle criteria using Lyapunov functions and then applies the circle criteria to various magnitudes of the feedback gain. For example, if G(s) of figure 5-1 is:

$$G(s) = \frac{-s + 1}{s^2 + 2s + 1} \qquad (1)$$

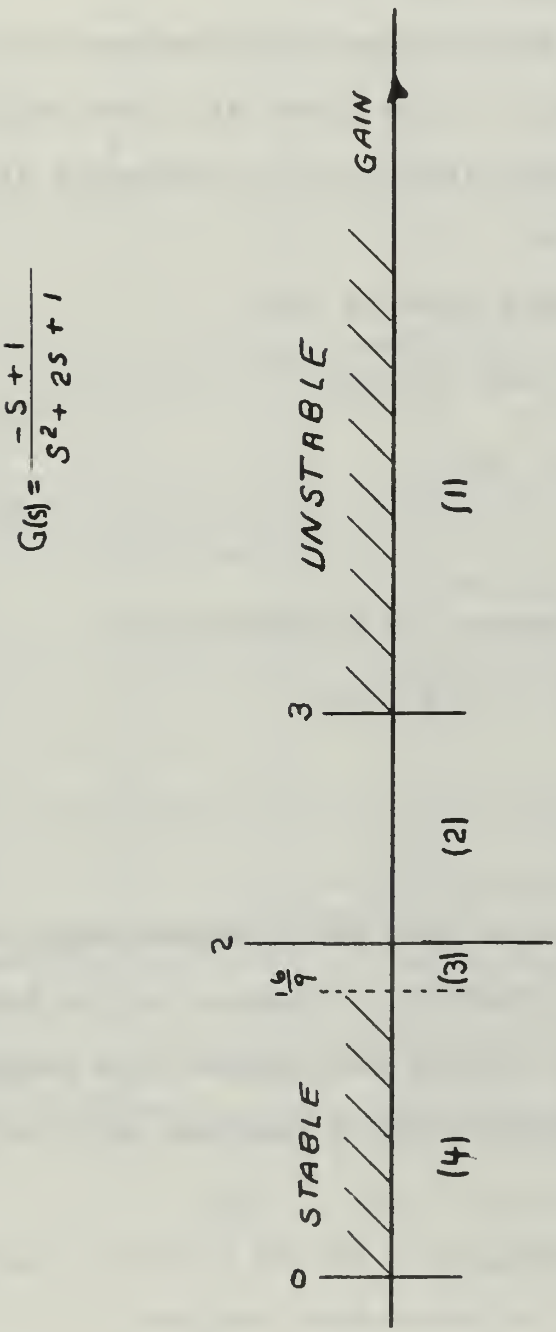
Then, the resulting stability/instability regions, for various magnitudes of the feedback gain; a(t), are shown in figure 5-4. The regions have the following significance:

a. The system is unstable provided the minimum value, α, and the maximum value, β, of the feedback gain, a(t), are in region one.

b. The system is unstable for β in region two provided:

$$\alpha > \frac{9\beta^2 - 16\beta}{3\beta^2 - 9\beta + 8 + 4(\beta-2)\sqrt{\beta+1}} \qquad (2)$$

Figure 5-4: Stability Regions for the Magnitude of a(t)



c. The system is ASIL for β in region three provided:

$$\alpha > \frac{9\beta^2 - 16\beta}{3\beta^2 - 9\beta + 8 - 4(\beta-2)\sqrt{\beta+1}} \qquad (3)$$

d. The system is ASIL provided both α and β are restricted to region four.

The limits of the various regions were determined by applying the circle criteria to the given G(s) and magnitudes of a(t). That the conditions are not necessary is shown in figures 5-5 and 5-6.

In figure 5-5, the solid line is for:

$$a(t) = 1. + \cos\ (0.15t) \qquad (4)$$

and the dotted line is for:

$$a(t) = 1. \qquad (5)$$

In figure 5-6, the response is compared for:

$$a(t) = 1. + 4.\ \cos\ (15t) \qquad (6)$$

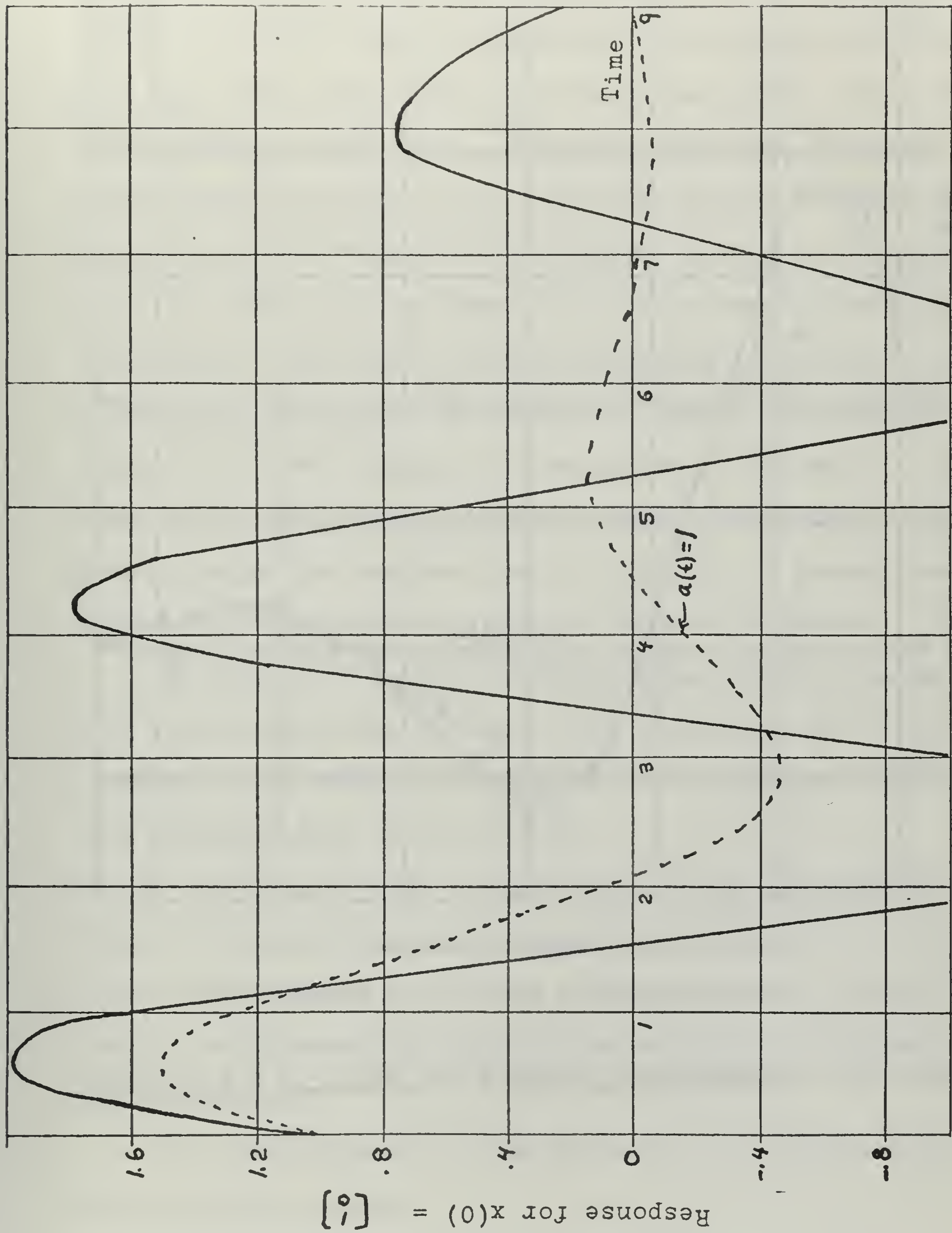
and:

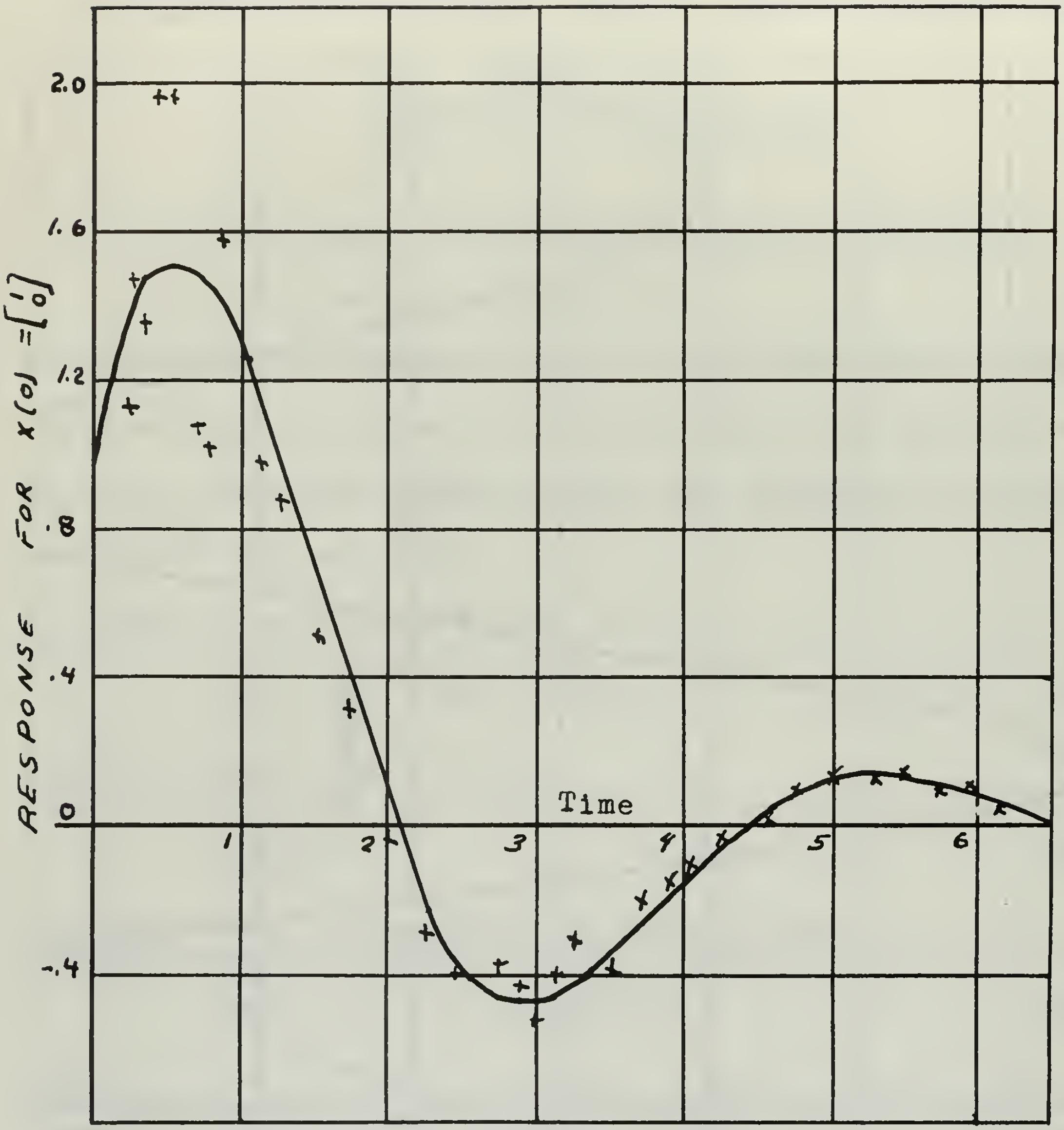
$$a(t) = 1. \qquad (7)$$

It is of interest to note that the response when the time variation is of high frequency (compared to the bandpass of G(s) of about four radians per second) the response is grouped around the response for a constant a(t) gain of one.

G. <u>Dual Mode Stability</u>. In the previous sections the current stability criteria were reviewed for the sys-

Figure 5-5: Comparative Response for a(t) = 1 + Cos(.15t)





Solid line: response for a(t) = 1

x: response for a(t) = 1 + 4 Cos(15t)

Figure 5-6: <u>Comparative Response for a(t) = 1 + 4 Cos(15t)</u>

tem of figure 5-1. As was noted above, it is always possible to provide system compensation to ensure stability for most given G(s) plants and known magnitude limits of the feedback gain, a(t). However, the phase and gain margins depend not only on the magnitude of the feedback gain but also on the shape of the feedback gain. In this section the effect of the shape of a(t) on stability will be considered. The basis of the discussion is that the state transition matrix has two stability modes: an unstable mode and a stable mode. In the system of figure 5-1, the mode of the state transition matrix is controlled by the magnitude of the feedback gain, a(t). It is convenient to define an instantaneous stability mode as follows:

Definition 5-1: The instantaneous stability mode of the state transition matrix is the stability of a corresponding time invariant system with a fixed gain equal to the instantaneous value of a(t).

It is conjectured that a comparison of the instantaneous stability modes, to determine whether the set of instantaneous stable modes or the set of instantaneous unstable modes predominate, will provide an estimate of system stability margins.

The significance of the definition is illustrated in the following example.

Example 5-1 : See figures 5-1 and 5-7.

Let:

$$G(s) = \frac{-s + 1}{s^2 + 2s + 1}$$

$$u(t) = 0$$

$$a(t) = \begin{cases} 0 & t \geq T \\ 3 & t \leq T \end{cases}$$

Since the selected G(s) is asymptotically stable in the large, the system will be ASIL for a(t) = 0. The stability limits of the system with fixed gain feedback, A, are determined by the characteristic equation:

$$1 + A\, G(s) = 0 \qquad (1)$$

where: $A \triangleq$ fixed feedback gain

The feedback gain, a(t), and stability limits for this example are shown in figure 5-7. For the given G(s), a fixed system will be asymptotically stable in the large for gains between minus one and plus two. Hence, the system of this example will be instantaneous unstable during the period when a(t) equals three and instantaneous stable during the period when a(t) equals zero. Clearly, for any finite switching time, T, the instantaneous stable mode will predominate and the system will be asymptotically stable in the large.

It is of interest to carry the illustration a step further by considering gains of the form:

$$a(t) = A \sin \omega t \qquad (2)$$

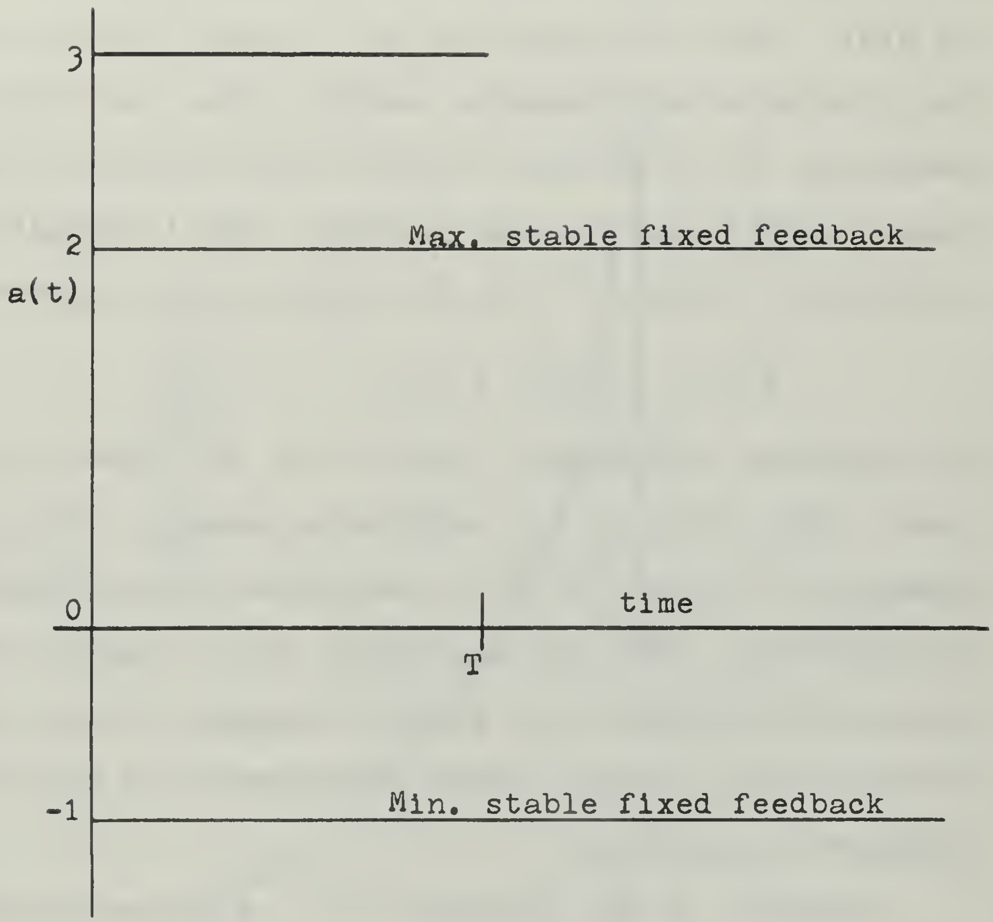


Figure 5-7: Figure for Example 5-1

In equation (2), the average value of a(t) is zero, hence; the fixed stability limits remain the same as in figure 5-7. The plant is a low pass filter. The magnitude of $G(j\omega)$ is shown in figure 5-8. The first set of responses, figure 5-9, a frequency was selected within the bandpass of G(s). When the magnitude, A, is less than one, the system is always instantaneous stable. When the value of the magnitude, A, is greater than one, the system is instantaneous unstable during some periods. The instability is indicated in figure 5-9 by the loops in the response. For:

$$a(t) = 3 \sin t \; ; \; \omega = 1 \tag{3}$$

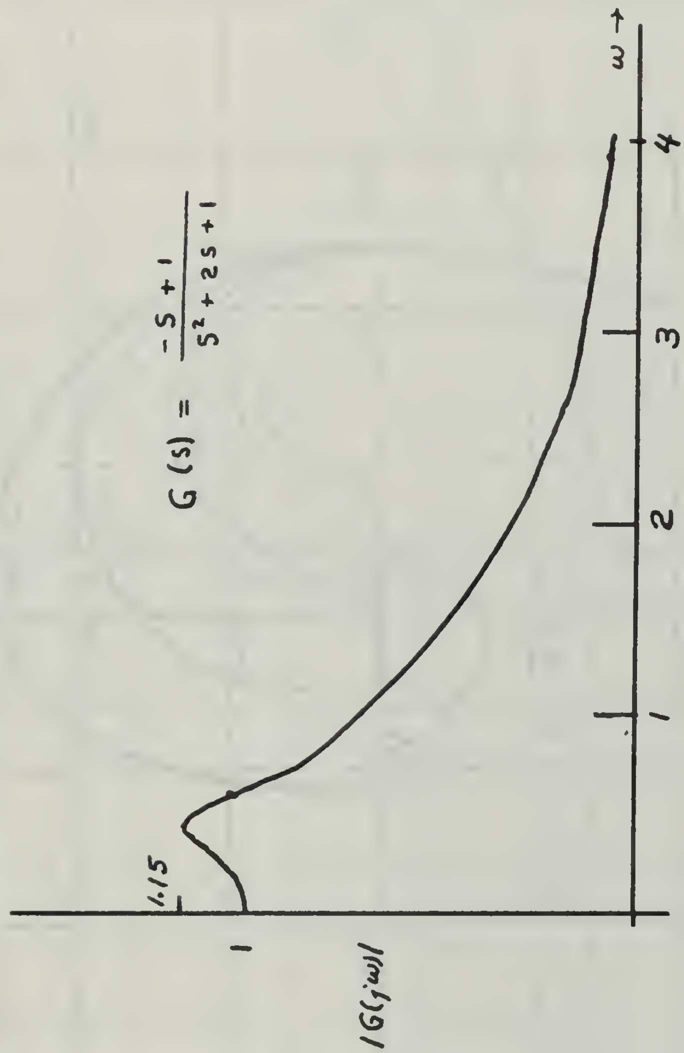
the response is bounded, figure 5-10, but does not go to zero. The effect of the modulation product, a(t)y(t), appears in figure 5-10 as a modulation of the steady state oscillations. When the magnitude; A; in equation (2) is greater than three for a radian frequency of one, the instantaneous unstable modes predominate and the overall response is unstable.

Finally, if the frequency; ω ; is chosen outside the bandpass of $G(j\omega)$, the instantaneous unstable mode has less affect on the system response. Figure 5-11 shows the response near the origin for:

$$a(t) = 2 \sin 10t \tag{4}$$

It is conjectured that, for the system of figure 5-1, a more exact criteria for system stability can be obtained in terms of the instantaneous stability modes and the fixed

Figure 5-8: Magnitude Response of G(s)



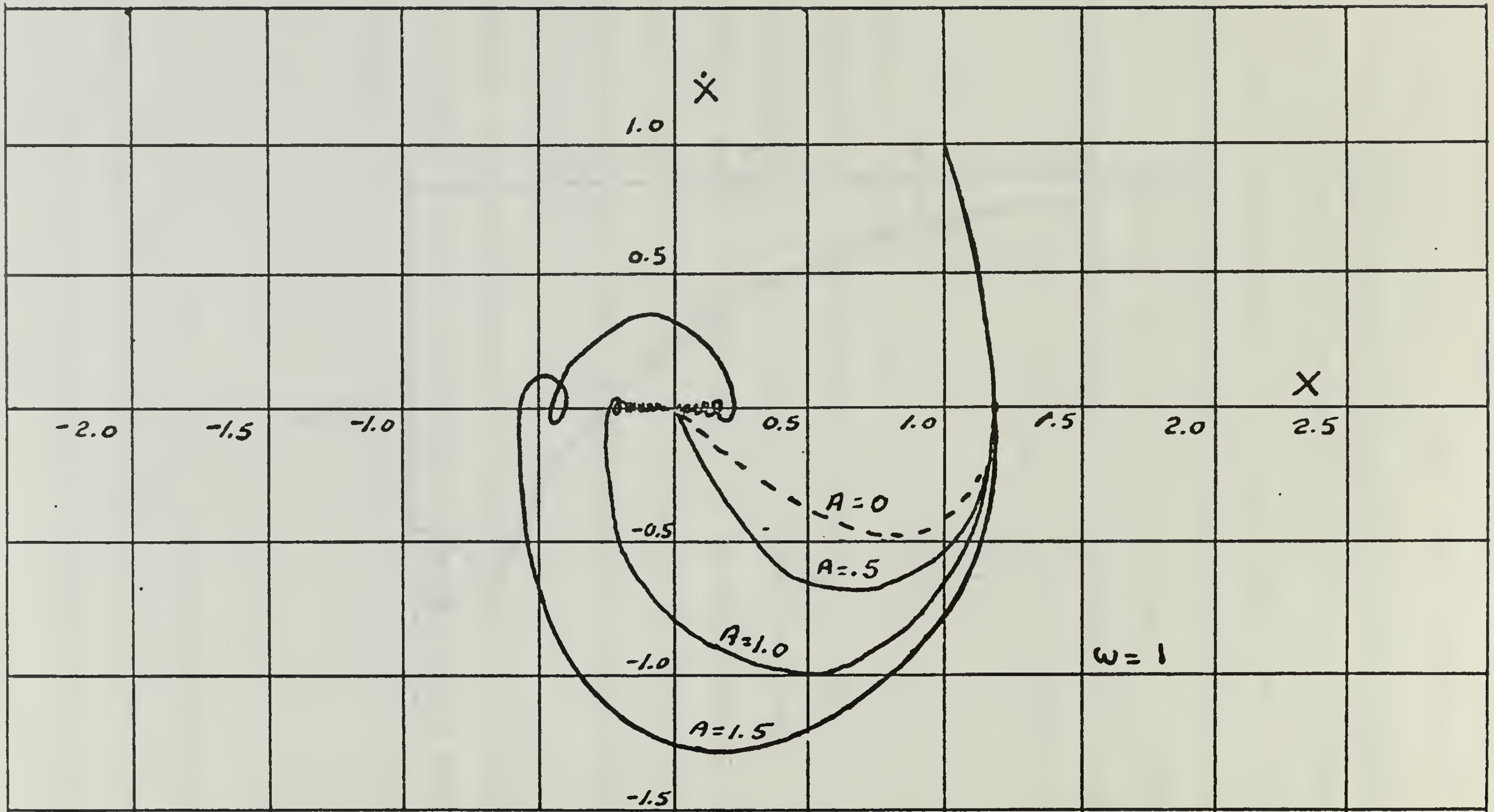
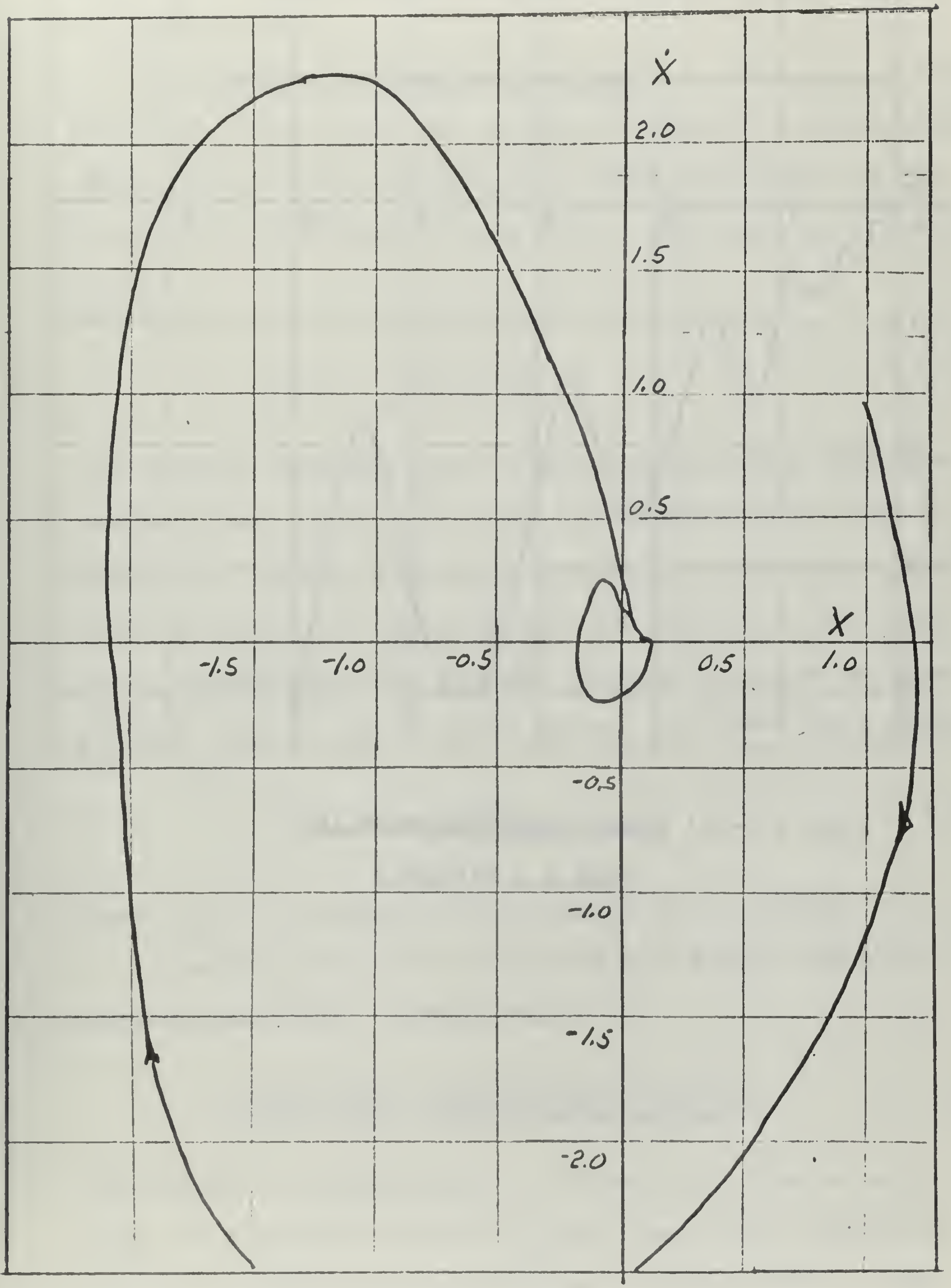


Figure 5-9:
<u>Phase Plane Response</u>

Figure 5-10: Phase Plane Response for a(t) = 3 Sin(t)



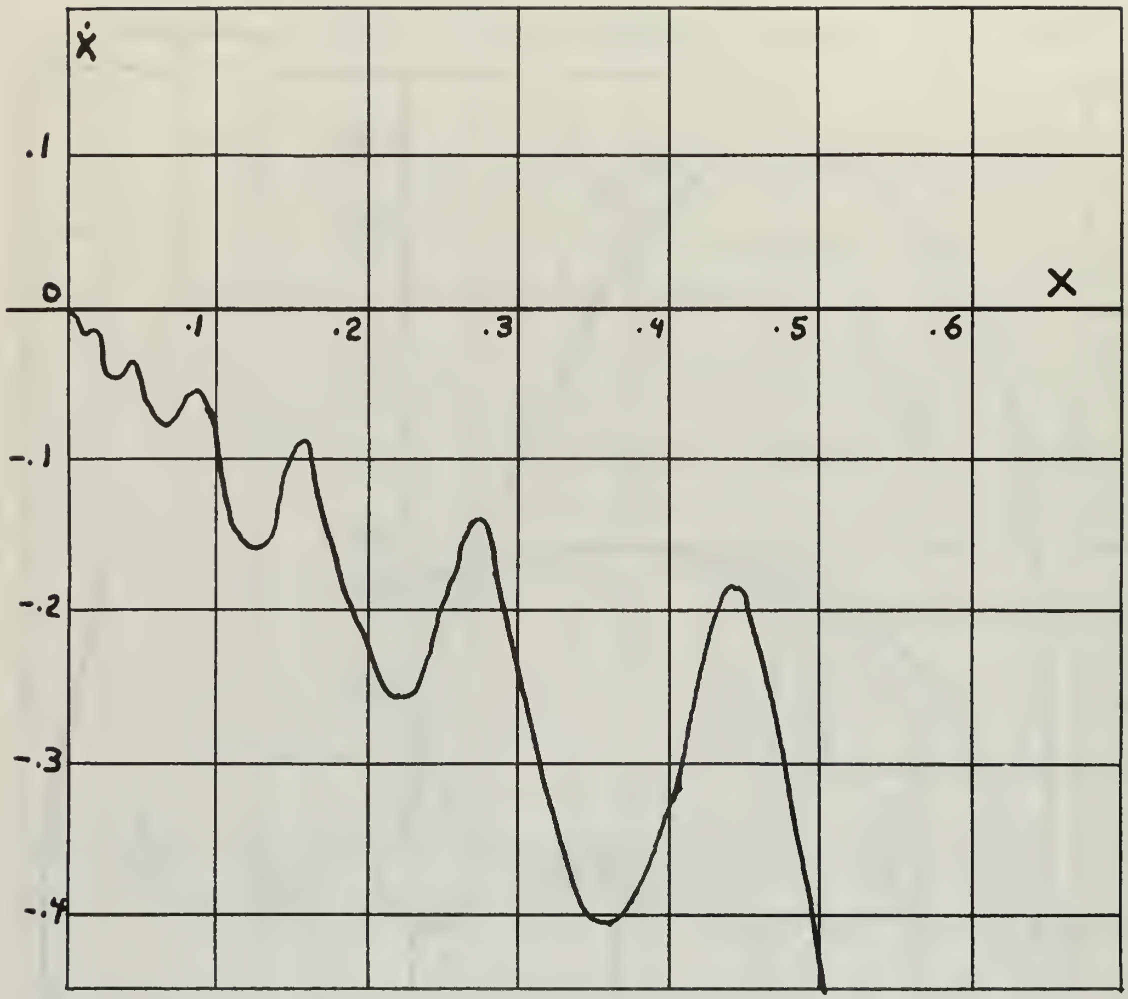


Figure 5-11: Phase Plane Response for

a(t) = 2 Sin(10t)

plant's bandpass then can be obtained by the circle criteria. Some results supporting this conjecture are given in the next section.

H. Dual Mode Stability Results. In this section some stability results for the system of figure 5-1 are presented in terms of the instantaneous stability modes and the bandpass for the fixed plant, G(s). The basic G(s) selected is:

$$G(s) = \frac{-s + 1}{s^2 + 2s + 1} \tag{1}$$

This G(s) is asymptotically stable in the large when the feedback gain, a(t), is zero. The feedback gain used to obtain the following results is assumed to have an average value of zero. If the average value of a(t) is not zero, it is assumed that the average value of a(t) can be incorporated into the fixed plant giving, as a new fixed plant:

$$G_1(s) = \frac{G(s)}{1 + \overline{a(t)}\, G(s)} \tag{2}$$

where $G_1(s)$ is asymptotically stable in the large.

The criteria on the instantaneous stable modes is a simple time ratio. Specifically:

$$S_u = \frac{\text{Total time instantaneous unstable}}{\text{Total time instantaneous stable}} \tag{3}$$

The method of determining the criteria, S_u, is to calculate the stability limits for the fixed plant (stability limits for equation (1) are given in figure 5-7). When the fixed plant stability, limits are known, the feedback gain; a(t); is plotted and the total time a(t) is outside the fixed plant limits is measured to obtain the total time the system is instantaneous unstable. The total time the system is instantaneous stable is determined by measuring the time a(t) is within the fixed plant stability limits. To simplify calculations, only periodic feedback gains were utilized.

To obtain the results shown in figure 5-12, feedback gains of the form:

$$a(t) = A \sin \omega t \tag{4}$$

or

$$a(t) = A \cos \omega t \tag{5}$$

are used. In terms of the stability ratio; S_u; the response to the gains of equation (4) and equation (5), resulted in three consistent regions. For values of S_u in region I the response was asymptotically stable. For values of S_u in region III the response was asymptotically unstable. Regions I and III are divided by a region (region II) of uncertainty. It is suspected that, with additional computer runs, the width of region II could be narrowed. It is significant that the stability regions do not depend on the constant phase component of a(t), but do depend on the frequency component of a(t).

A simple interpretation of the above result is that during the instantaneous unstable mode the norm of the state vector is moving away from zero. During the instan-

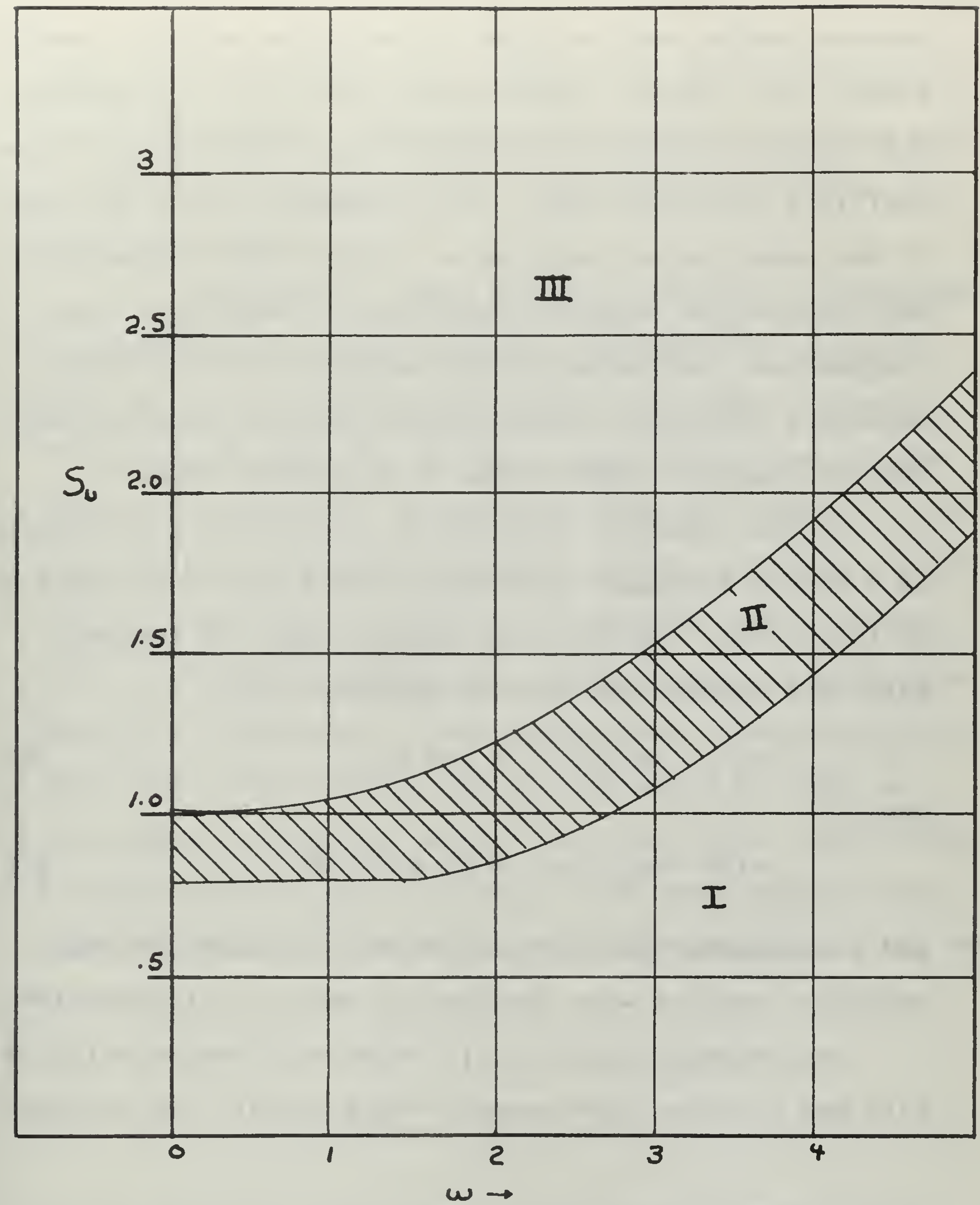


Figure 5-12: Dual Mode Stability Regions

taneous stable mode the norm of the state vector is moving toward zero. Hence, the stability criteria, S_u, measures the relative change in the norm of the state vector. Since G(s) is a low pass filter, it is reasonable that the norm of the state vector requires a finite time to change (as does the voltage across a capacitor). Hence, for high frequencies, the change in the norm of the state vector occurring during the instantaneous unstable mode becomes smaller, and the stable value of S_u becomes larger.

If the stability criteria; S_u; held only for feedback of a single frequency, the regions of figure 5-12 would be of little use. To test more general types of feedback gain, the stable regions were obtained for:

$$a(t) = A\ (\sin t + \cos t) \tag{6}$$

and:

$$a(t) + A_1 \sin t + A_2 \sin 1.5t \tag{7}$$

and a triangular gain of period two. In each case the stability regions were the same as shown in figure 5-12*.

For feedback gains; a(t); where the average value of a(t) was not zero, the avergae value of a(t) was incorpo-

---

* In the case of several frequency components, S_u, was determined for the overall waveform and plotted for the lowest frequency component. This is one of the reasons for the width of region II.

rated into the fixed plant (equation 2) and new stability bounds determined. Provided that the resulting fixed plant is asymptotically stable, the same stability regions, figure 5-12, apply.

I. Conclusion. For a sepcific system it is possible to determine the system response and stability using digital solutions of the system state equations. However, this approach does not provide a clear indication of the limits of stability. Attempts to study the limits of stability using frequency techniques (circle criteria) result only in sufficient conditions and provide little information on margins of stability. In an attempt to study the margins of stability, it was conjectured a stability criteria (for some systems) could be obtained in terms of dual modes of response (stable and unstable). Results for a simple criteria indicate that a dual mode approach may offer a means of refining the sufficient conditions for stability.

6. CONCLUSIONS

One of the traditional methods of solving time invariant system equations is to transform the equations into either Laplace or Fourier algebric equations. Application of these standard transforms to time varying system equations normally results in a convolution integral equation instead of an algebric equation. However, it is possible to generalize the transform process to give algebric equations for specific systems.

Additionally, it is possible to define time varying system functions in terms of the Laplace and Fourier variables. These time varying system functions may be used to determine the response of certain systems. However, the general usefulness of the transform approach to time varying is limited by:

a. The limitation of compatible kernals to systems similar to the system for which the kernal for derived,

b. The difficulty of generating a compatible kernal,

c. The fact that if the kernal is chosen for conceptual convenience, the resulting system function for interconnected blocks will not generally be algebric.

The state variable method for solving time invariant systems can be extended to time varying systems. However, the solution of the state equations is more difficult since an exponential series expansion for the state transition matrix is not always possible. Despite the additional problemsof determining the state transition matrix, state variable methods are more useful and direct than the transform methods. Additional research is needed in the methods of determining the state transition matrix.

Certain time varying systems, i.e. periodic, are equivalent to some time invariant systems. If the equivalent time invariant system can be determined, the system response and stability can be obtained with standard time invariant techniques. Since an equivalent time invariant system exists for each periodic time varying system, the analysis of these systems would be simplified if convenient methods could be found for obtaining the required linear transformation matrix.

The special case of linear time varying feedback systems, where the time variation is only in the feedback loop, requires the establishment of stability criteria which are more than sufficient. The present sufficient conditions for stability preclude the specification of gain and phase margins. One approach that may be of use in refining the sufficient conditions for stability, is to consider the system as a dual mode system. In one mode the system response is stable. In the other mode the system response is unstable. Then the system is either stable or unstable depending upon which mode predominates. The problem with this approach is the selection of suitable criteria for determining which mode predominates. In this thesis a simple time ratio criteria was adopted. Application of this criteria to a specific system has shown that a refinement of the sufficient conditions for stability was possible. Further research is desirable to determine if the selected criteria can be applied to arbitrary systems. If a simple dual mode stability criteria, $S_u(\omega)$, can be obtained for arbitrary systems, then compensation based on phase margins and gain margins becomes possible.

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APPENDIX

Solution of State Equations.

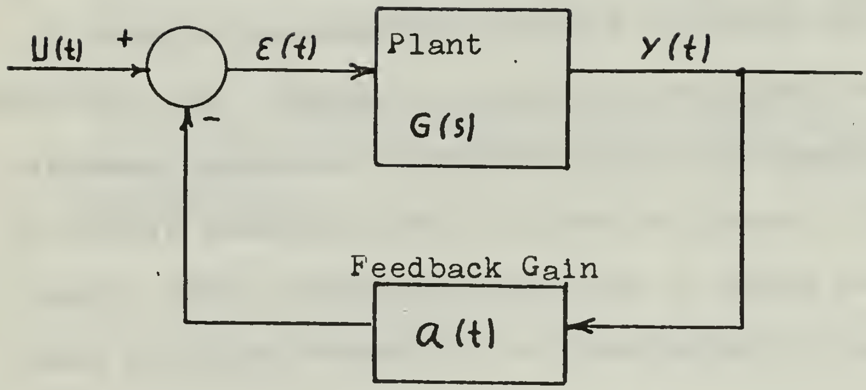
The solution of the system, shown in figure A-1, is obtainable by either a Runge-Kutta or a difference equation method. The Runge-Kutta sub-routine unsed provides the state Equation using a fourth order approximation. In the difference equation solution, advantage is taken of the fact that the open loop plant is fixed. For the fixed plant an exponential series expansion is always possible for the state transition matrix. The feedback signal is treated as an input to the open loop plant, G(s). The block diagram illustrating the difference equation solution is shown in figure A-2.

There is a difference in the numerical results of the two solutions. The difference results from three effects:

a. Approximation errors in the Runge-Kutta solution [17],

b. Approximation errors in the difference equation solution. In the difference equation solution, the errors are mainly in the subroutine that provides the state transition matrix and the input transition matrix (Gamma).

c. The fact that the difference equation solution uses a single time interval for each step while the Runge-Kutta solution approximates the transition by a series of polynominal interpolations [17].



$$G(s) = \frac{Q(s)}{P(s)}$$

Where: u(t) is the system input

ε(t) is the error function

G(s) is a fixed parameter plant

y(t) is the system output

a(t) is the linear time varying gain

Q(s) is the numerator polynominal of G(s)

P(s) is the denominator polynominal of G(s)

Figure A-1: <u>Example of a Simple Time Varying Feedback System</u>

The effect of these differences is difficult to specify. However, the results can be compared for a specific example. Figure A-3 shows the difference in the two solutions for a second order plant where:

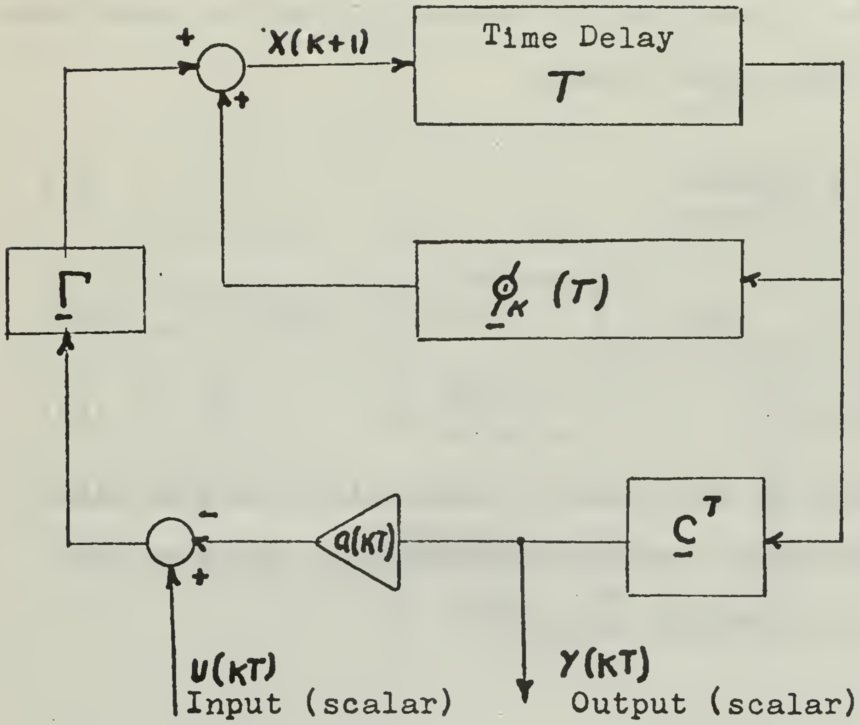
$$G(s) = \frac{-s+1}{s^2+2s+1} \tag{1}$$

and:

$$a(t) = 1 \tag{2}$$

The significance of the result, figure A-3, is the existence of a difference between the solutions whenever the state vector is changing rapidly.

Figure A-2: Difference Equation Solution



For the system of figure A-1:

$$\dot{x}(t) = \underline{A}\,x(t) + \underline{B}\left(-a(t)y(t) + u(t)\right)$$

$$y(t) = \underline{c}^T x(t)$$

Where:

$$Q(s) = q_m s^m + \cdots + q_1 s + q_0$$

$$P(s) = s^n + \cdots + p_1 s + p_0$$

(The explanation of the terms on this figure is continued on the next page)

Figure A-2a: Difference Equation Solution

$$\underline{A} = \begin{bmatrix} 0 & \vdots & \underline{I}_{N-2} \\ \cdots & \cdots & \cdots \\ -P_0, & \cdots & -P_{N-1} \end{bmatrix}$$

$$\underline{C}^T = [g_0, g_1, \cdots g_m]$$

$$\underline{B} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

Let: $\epsilon(t) = -a(t)y(t) + u(t)$ (a scalar)

Then the difference equations are:

$$X[(K+1)T] = \underline{\phi}_K(T)X(KT) + \Gamma(T)\epsilon(KT)$$

Where:

$$\underline{\phi}_K(T) = EXP[\underline{A}T]$$

$$\underline{\Gamma}(T) = \int_0^T EXP[\underline{A}t - \underline{A}\tau]\,\underline{B}\,U(\tau)\,d\tau$$

Note: $\phi_K(T)$ and $\Gamma(T)$ were obtained by an exponential series expansion.

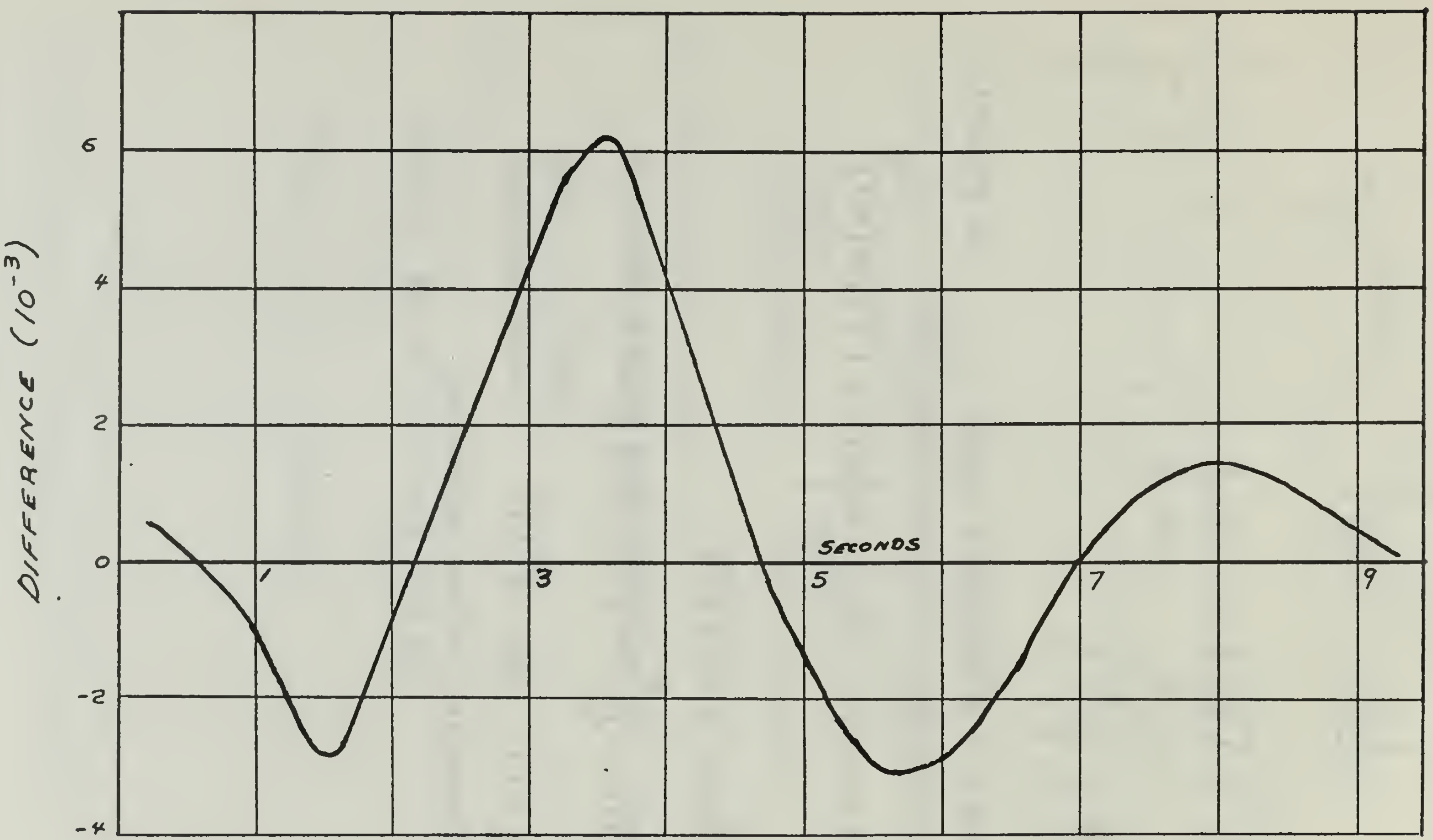


Figure A-3: Difference Between a Runge-Kutta and a Difference Equation Solution for constant Unity Gain Feedback

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DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

METHODS OF ANALYSIS FOR LINEAR SYSTEMS WITH TIME VARYING PARAMETERS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

M. S. Thesis - Electrical Engineering - 1968

5. AUTHOR(S) (Last name, first name, initial)

MACDOUGALL, Donald G.
Lieutenant, United States Navy

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1968	103	20

8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. AVAILABILITY/LIMITATION NOTICES

[redacted]

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

This thesis presents a comparative study of frequency domain and state variable (time domain) methods for the solution of linear time varying systems. Stability criteria for linear feedback systems, where the time variation is limited to the feedback loop, are considered. In comparing state variable and frequency domain solutions, the conclusion is reached that state variable methods are more useful. Experimental computer results are presented which indicate that, for the feedback system considered, sufficient conditions for stability can be refined by considering the system to be dual mode (a stable mode and an unstable mode), and by considering which mode predominates.

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Time Varying Circuits						
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Generalized Transforms						
System Functions						
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Stability Criteria						

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